Strongly Bounded, Finitely Additive Vector Measures And Weak Sequential Compactness

By

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This dissertation is concerned primarily with the condition of uniform strong boundedness on a set K of vector-valued, finitely additive measures defined on a ring. K is uniformly strongly bounded if the images of disjoint sequence of elements of the ring converge to zero uniformly in the measures of K. If K consists of one measure, the measure is called strongly bounded. We give an elementary proof of a well known result which, when the measures are countably additive on a sigma-algebra, frequently allows the reduction of a problem in vector measure theory to classical (non-negative) measure theory. Using strong boundedness, we generalize this result to finitely additive, vector-valued measures on rings. An extension property which we develop enables us to derive relations between uniform strong boundedness and absolute continuity with respect to a non-negative measure. We apply these results to obtain equivalent conditions for weak sequential compactness in spaces of bounded, finitely additive, scalar-valued measures defined on an algebra, including easy proofs of several well known conditions equivalent to weak sequential convergence.

INTRODUCTION

In developing a theory of integration with countably additive vector measures, Bartle, Dunford, and Schwartz [1] found that a vector measure μ had associated with it a non-negative measure λ such that λ is a "control measure" for μ in the sense that λ and μ have the same class of null sets. This enabled them in many instances to transfer a problem in vector measure theory to classical measure theory. We give a short, elementary proof of the Bartle-Dunford-Schwartz Theorem in Chapter 1 (see page 28). In the second chapter we consider the problem of generalizing the Bartle-Dunford-Schwartz result to finitely additive measures. In fact, we show that such a λ exists for a given finitely additive vector measure μ on a ring precisely when μ is strongly bounded. Strongly bounded measures were introduced by Rickart [14], where it was shown that every such measure is bounded. We shall see that for weakly sequentially complete vector spaces, the converse is also true; that is, any bounded measure is also strongly bounded. Thus this condition on finitely additive measures is at once powerful in its implications and mild in its restrictions. The path we shall follow investigates the question of extending countably additive, strongly bounded measures on a ring to countably additive measures on the generated

σ-ring, and we are able to strengthen an extension theorem of Sion [15]. Similar extension problems were considered by Dinculeanu and Kluvanek [5], and several problems raised in [5] will be solved in 2.20 and 2.27 (in the bounded measure case). We shall use these results in Chapter 3 to obtain powerful results on sets of uniformly strongly bounded vector measures. Specializing to the case of scalar-valued measures, we obtain various useful characterizations of weak sequential compactness in spaces of bounded, finitely additive scalar measures on algebras. These characterizations complement those obtained by Dunford and Schwartz [6], and are related to ones obtained by Porcelli [13]. Earlier, Grothendieck [8] had obtained similar results on weak sequential compactness for Radon measures on σ-algebras. Finally, Darst [4] was able to simplify the theory by giving a more direct proof of Porcelli's condition for weak sequential convergence. We are able, using the idea of uniform strong boundedness, to give a short proof of the equivalence of Porcelli's condition to a related one of Hildebrandt [10], and using our earlier theorems, an easy proof of the equivalence of either of these to weak sequential convergence.

Similar results on "control measures" and on uniform strong boundedness have been obtained by Brooks [2]. However,

the related theorems were arrived at independently, and the methods of proof in this dissertation are entirely different. In particular, Theorem 3.6 is a strengthening of a result of Brooks [2].

CHAPTER I

Basic Results

1.1 We shall be concerned in this work with various conditions concerning vector measures. This chapter will lay the foundation for the sequel. Theorems 1.12 and 1.15 will be of particular importance in Chapter 2.

1.2 Notation. The conventions established in this section will be followed throughout the dissertation.

E denotes a locally convex linear topological space with completion $\tilde{E}$. If p is a continuous semi-norm on E, the unique continuous semi-norm on $\tilde{E}$ which extends p is denoted by $\tilde{p}$. E' is the continuous dual of E. The topology induced on E by E' is called the weak topology. The scalar field of E is Φ, and Φ may be real or complex. The real numbers are denoted by $\mathbb{R}$, the non-negative real numbers by $\mathbb{R}^+$, $\mathbb{R}^* = \mathbb{R}^+ \cup \{\infty\}$, and the non-negative integers are denoted by ω. For $U \subseteq E$, U is balanced if $aU = \{au: u \in U\} \subseteq U$ whenever $|a| \leq 1$, $a \in \Phi$. We denote n-dimensional Euclidian space by $\mathbb{R}^n$, or if $n = 1$ simply by $\mathbb{R}$. When $E = \mathbb{R}^n$, Φ is of course always $\mathbb{R}$. If $U \leq E$ and if p is a continuous semi-norm, we say U is p-bounded if $p(U) = \{p(u): u \in U\}$ is bounded in Φ. U is bounded if and only if, for each continuous semi-norm p, U is

p-bounded. We write l.c.s. for locally convex linear topological space. A standard reference for theorems in a l.c.s. is Köthe [11]. E will always be Hausdorff; we will write $\mathfrak{X}$ instead of E when E is a Banach space.

A ring R of subsets of a space Ω is a set of subsets of Ω closed under finite unions and differences of elements of R. R will denote a ring unless otherwise specified. If Ω is required to be an element of R, we will write A instead of R and call A an algebra. If $r\in R$, the algebra $R_{/r} = \{r'\cap r: r'\in R\}$ is the algebra induced by R on r. A δ-ring D is a ring closed under countable intersections, and a σ-ring $\sum$ is a ring closed under countable unions. The σ-ring generated by a ring R is $\sum(R)$. A sequence $(r_i)_{i\in\omega} \subseteq R$ is monotone increasing to $r = \cup_i r_i$ ($r_i \uparrow r$) if $r_i \subseteq r_{i+1}$, $i\in\omega$; $(r_i)_i$ is monotone decreasing to $r = \cap_i r_i$ ($r_i \downarrow r$) if $r_i \supseteq r_{i+1}$, $i\in\omega$. Any function μ on a ring R into a l.c.s. E is a measure. A measure $\mu: R \to E$ is countably additive if for every disjoint sequence $(r_i) \subseteq R$, $\cup_i r_i \in R$ implies

$$\mu(\cup_i r_i) = \sum_i \mu(r_i),$$

where the convergence is unconditional. The measure μ is finitely additive if the above equality holds for every finite disjoint family $(r_i)_{i=1}^n \subseteq R$. If $(\mu_\alpha)_{\alpha\in\mathfrak{A}}$ is a set

of measures, (μ_α) is uniformly bounded if

$$\{\mu_\alpha(r): \alpha\in\mathfrak{A}, r\in R\}$$

is bounded in E; if (μ_α) is just a single measure μ, we say that μ is bounded. The set $(\mu_\alpha)_\alpha$ is point-wise bounded if for each $r\in R$, $\{\mu_\alpha(r): \alpha\in\mathfrak{A}\}$ is bounded. For p a continuous semi-norm on E, the p-semi-variation, $p(\mu): R \to \mathfrak{R}^*$, is defined by

$$p(\mu)(r) = \sup\{p(\mu(r')): r'\subseteq r, r'\in R\}, \qquad r\in R.$$

If E is a normed linear space with norm $||\ \ ||$, the semi-variation of μ, $||\mu||\ (\cdot)$ is simply the $||\ \ ||$-semi-variation of μ. Also, $\nu(\mu)(\cdot)$ is the total variation of μ, defined when E is a normed linear space by

$$\nu(\mu)(r)=\sup\{\sum_{i=1}^{n}||\mu(r_i)||: (r_i)_{i=1}^{n}\subseteq R, (r_i) \text{ disjoint, and } \cup_i r_i\supseteq r\}.$$

If p is a continuous semi-norm on E, and $\lambda: R \to \mathfrak{R}^+$ is finitely additive, we write $\mu <<_p \lambda$ when

$$\lim_{\lambda(r)\to 0} p(\mu(r)) = 0,$$

that is, given $\epsilon > 0$ there exists a $\delta > 0$ such that for all $r\in R$ such that $\lambda(r) < \delta$, we have $p(\mu(r)) < \epsilon$. In a normed

space, we will omit the "p" and write simply $\mu << \lambda$. For $x \in E'$, we say $|x| \leq p$ if $|x(e)| \leq p(e)$, $e \in E$. We will indicate the end of a proof with $\square$.

If $\mu: R \to \mathfrak{R}$ is a finitely additive measure, then we define $\mu^+: R \to \mathfrak{R}^*$ and $\mu^-: R \to \mathfrak{R}^*$ by

$$\mu^+(r) = \sup\{\mu(r'): r' \in R, r' \subseteq r\},$$

$$\mu^-(r) = -\inf\{\mu(r'): r' \in R, r' \subseteq r\}.$$

As we shall see in Theorem 1.3, if μ is a bounded measure, we can decompose μ as the sum of two non-negative, bounded measures by $\mu = \mu^+ - \mu^-$.

1.3 Theorem. Let $\mu: R \to \mathfrak{R}$ be a finitely additive measure. Then

(1) μ^+ and μ^- are non-negative finitely additive measures;

(2) $\nu(\mu)(r) = \mu^+(r) + \mu^-(r)$, $r \in R$;

(3) $|\mu|(r) = \max\{\mu^+(r), \mu^-(r)\}$, $r \in R$.

If μ is point-wise bounded, we have

(4) $\mu = \mu^+ - \mu^-$.

Proof (1) μ^+ and μ^- are obviously non-negative.

Let r,s be disjoint elements of R. Then

$$\mu^+(r\cup s) = \sup\{\mu(t): t\in R,\ t\subseteq r\cup s\}$$

$$= \sup\{\mu(t_1\cup t_2): t_1,\ t_2\in R,\ t_1\subseteq r,\ t_2\subseteq s\}$$

$$= \sup\{\mu(t_1) + \mu(t_2): t_1, t_2\in R,\ t_1\subseteq r,\ t_2\subseteq s\}$$

$$= \mu^+(r) + \mu^+(s).$$

Thus μ^+ is finitely additive. Since $\mu^- = (-\mu)^+$, μ^- is also finitely additive.

(2) Set $r\in R$. Then if $R' = R/_r$

$$\nu(\mu)(r) = \sup\{\sum_{i=1}^{n} |\mu(r_i)|: n\in\omega,\ (r_i) \text{ disjoint in } R\}$$

$$= \sup\{\mu(s)-\mu(t): s,t\subseteq r,\ s,t\in R,\ s\cap t=\phi,\ \mu(s)\geq 0\geq\mu(t)\}$$

$$= \sup\{\mu(s)-\mu(t): s,t\subseteq r,\ s,t\in R,\ s\cap t=\phi\}$$

$$= \sup\{\mu(s)-\mu(t): s,t\subseteq r,\ s,t\in R\}$$

$$= \mu^+(r) + \mu^-(r).$$

The third equality above follows from: $\mu(s), \mu(t) \geq 0$ imply $\mu(s) - \mu(t) \leq \mu(s) = \mu(s) + \mu(\phi)$; $\mu(s), \mu(t) \leq 0$ imply $\mu(s) - \mu(t) \leq -\mu(t) = \mu(\phi) - \mu(t)$; $\mu(t) \geq 0$ and $\mu(s) \leq 0$ imply $\mu(s) - \mu(t) \leq \mu(\phi) - \mu(b)$. The fourth equality follows from: $\mu(s\cap t) \leq 0$ implies $\mu(s) - \mu(t) \leq \mu(s-t) - \mu(t)$; $\mu(s\cap t) \geq 0$ implies $\mu(s) - \mu(t) \leq \mu(s) - \mu(t-s)$.

(3) $|\mu|(r) = \sup\{|\mu(s)| : s \subseteq r,\ s \in R\}$

$= \max\{\sup\{\mu(s) : s \subseteq r, s \in R, \mu(s) \geq 0\}, \sup\{-\mu(s) : s \subseteq r, s \in R, \mu(s) \leq 0\}\}$

$= \max\{\mu^+(r),\ \mu^-(r)\}$.

(4) Let $r \in R$. Then

$$\mu(r) + \mu^-(r) = \mu(r) + \sup\{-\mu(s) : s \subseteq r,\ s \in R\}$$

$$= \sup\{\mu(r) - \mu(s) : s \subseteq r,\ s \in R\}$$

$$= \sup\{\mu(r-s) : s \subseteq r,\ s \in R\}$$

$$= \mu^+(r).$$

Since μ is point-wise bounded, $\mu^-(r)$ is finite. Thus $\mu = \mu^+ - \mu^-$. □

1.4 Corollary. (1) $\nu(\mu): R \to \mathfrak{R}^*$ is finitely additive;

(2) μ is bounded if and only if $\mu^+(\mu^-)$ is bounded;

(3) $\mu^+(r) = \infty$ if and only if $\mu^-(r) = \infty$, $r \in R$.

Proof (1) above follows from (1) and (2) of Theorem 1.3; (2) follows from (3) of Theorem 1.3. Since by the proof of (4) of Theorem 1.3, $\mu + \mu^- = \mu^+$, (3) above follows from the fact that μ is finite-valued. □

1.5 Theorem. Let $\mu: R \to \mathfrak{R}^+$ be a bounded, countably additive measure and $\sum = \sum(R)$. For $s \in \sum$, we define

$$\bar{\mu}(s) = \inf\{\sum_i \mu(r_i) : (r_i) \subseteq R,\ \cup_i r_i \supseteq s\}.$$

Then $\bar{\mu}$ is a bounded, countably additive measure on $\sum$ into $\mathbb{R}^+$.

Proof It is well known that $\bar{\mu}$ defined as above is a countably additive extension of μ. Details can be found in Halmos [9]. Suppose $\bar{\mu}$ is not bounded. Then there exists $(s_i) \subseteq \sum$ such that

$$\bar{\mu}(s_i) > i, \qquad i \in \omega.$$

By the definition of $\bar{\mu}$, for $i \in \omega$, there is an $(r_{ij})_{j \in \omega} \subseteq R$ such that for some $N_i \in \omega$,

$$\sum_{j=1}^{N_i} \mu(r_{ij}) > i.$$

Defining $r_i = \cup_{j=1}^{N_i} r_{ij}$, we have

$$\mu(\cup_{j=1}^{i} r_i) > i, \qquad i \in \omega,$$

so that μ is not bounded on R. □

1.6 Corollary. In the definition of $\bar{\mu}$ above, the (r_i) may be required to be pair-wise-disjoint.

Proof Fix $s \in \sum$ and $(r_i) \subseteq R$ such that $\cup_i r_i \supseteq s$. Defining $s_i = r_i - \cup_{j=1}^{i-1} r_j$, we have $(s_i) \subseteq R$ pair-wise-disjoint, $\cup_i s_i \supseteq s$ and

$$\sum_i \mu(s_i) = \sum_i \mu(r_i - \cup_{j=1}^{i-1} r_j)$$

$$\leq \sum_i \mu(r_i).$$

Thus $\bar{\mu}(s)$ is equal to

$$\inf\{\sum_i \mu(r_i) : \text{disjoint } (r_i) \subseteq R;\ \cup_i r_i \supseteq s\}. \quad \square$$

The next theorem and Theorem 1.9 are basic for our work with finitely additive measures. They frequently allow us to reduce a problem to the case of a countably additive measure. Theorem 1.7 can be found in Dunford and Schwartz [6], although the proof there is stated for an algebra rather than a ring.

1.7 Theorem. For any ring R on Ω, there exists a ring R_s of open and closed subsets of a compact, Hausdorff space S such that R is isomorphic as a ring to R_s.

Remark: We call R_s the Stone ring of R, and the isomorphism from R onto R_s is denoted by τ.

Proof Let A be the algebra generated by $R \cup \{\Omega\}$. Then by Theorems IV 6.18 and IV 6.20 in Dunford and Schwartz [6], there is a compact Hausdorff space S such that $B(\Omega, A)$ is algebraicly, isometrically isomorphic to C(S) under a mapping h. Suppose $a \in A$. Let ξ_a denote the characteristic function of a. Then $h(\xi_a)$ is a continuous

function on S, while

$$h(\xi_a) = h(\xi_a^2) = h^2(\xi_a)$$

implies that $h(\xi_a) \subseteq \{0,1\}$. If we define

$$\tau(a) = [h(\xi_a)]^{-1}(1),$$

then by the continuity of $h(\xi_a)$ on S, $\tau(a)$ is an open and closed subset of S. We shall show that τ is an isomorphism, and define $R_s = \tau(R)$. Since h is one-to-one, so is τ. Set $a \in A$. Then

$$\begin{aligned}
\tau(\Omega-a) &= [h(\xi_{\Omega-a})]^{-1}(1) \\
&= [h(\xi_\Omega - \xi_a)]^{-1}(1) \\
&= [h(\xi_\Omega) - h(\xi_a)]^{-1}(1) \\
&= [1 - h(\xi_a)]^{-1}(1) \\
&= S - \tau(a),
\end{aligned}$$

since h being an algebraic isomorphism implies that h must send the multiplicative identity of $B(\Omega,A)$ onto the multiplicative identity of C(S). Similarly,

$$\tau(a_1 \cap a_2) = \tau(a_1) \cap \tau(a_2)$$

and

$$\tau(a_1 \cup a_2) = \tau(a_1) \cup \tau(a_2)$$

follow, respectively, from

$$h(\xi_{a_1 \cap a_2}) = h(\xi_{a_1} \cdot \xi_{a_2}) = h(\xi_{a_1}) \cdot h(\xi_{a_2})$$

and

$$h(\xi_{a_1 \cup a_2}) = h(\xi_{a_1} + \xi_{a_2} - \xi_{a_1 \cap a_2})$$

$$= h(\xi_{a_1}) + h(\xi_{a_2}) - h(\xi_{a_1}) \cdot h(\xi_{a_2}).$$

Suppose $a_1 \subseteq a_2$. Then $\tau(a_1) \subseteq \tau(a_2)$ since

$$\tau(a_1) = \tau(a_1 \cap a_2) = \tau(a_1) \cap \tau(a_2).$$

Then for $a_1, a_2 \in A$,

$$\tau(a_1 - a_2) = \tau(a_1) - \tau(a_1 \cap a_2)$$

$$= \tau(a_1) - (\tau(a_1 \cap a_2) \cup \tau(a_2 - a_1))$$

$$= \tau(a_1) - \tau(a_2),$$

since $a_2 - a_1 \subseteq \Omega - (a_1 - a_2)$ implies

$$\tau(a_2 - a_1) \subseteq S - \tau(a_1 - a_2).$$

Thus τ is a ring isomorphism, so that R_s is the desired ring. □

1.8 Remark. If $(s_i) \subseteq R$ is a disjoint sequence of non-null sets, then $\cup_i \tau(s_i) \notin R_s$.

Proof of Remark Suppose $(s_i) \subseteq R$ such that

$$\cup_{i=1}^{N} s_i \neq \cup_i s_i, \qquad N \in \omega.$$

Then $\cup_i \tau(s_i)$ is not compact, and hence not in R_s, since $(\tau(s_i))_i$ is an open cover of $\cup_i \tau(s_i)$ which has no finite subcover. In fact, if for some $N \in \omega$,

$$\cup_{i=1}^{N} \tau(s_i) = \cup_i \tau(s_i),$$

then for $i \in \omega$,

$$\tau(s_i) \subseteq \cup_{j=1}^{N} \tau(s_j),$$

implies

$$s_i \subseteq \cup_{j=1}^{N} s_j,$$

as in the proof of Theorem 1.7. □

One is tempted to think that Theorem 1.9 eliminates any problem in dealing with finitely additive measures. However, many results are valid only on σ-rings, and as Remark 1.8 shows, R_s is rarely a σ-ring. We shall show later that every ring contains a sequence (s_i) of disjoint non-null sets, unless R is a finite set. Thus R_s is a σ-ring if and only if R is finite.

1.9 Theorem. For every $\mu: R \to E$, the equation

$$T\mu(r) = \mu(\tau^{-1}(r)), \qquad r \in R_s,$$

defines a measure $T\mu: R_s \to E$. The mapping T has the following properties.

(1) For each finitely additive μ, $T\mu$ is countably additive;

(2) for each $(r_i) \subseteq R$, $\lim_i T\mu(\tau(r))$ exists in $\tilde{E}$ if and only if $\lim_i \mu(r)$ exists in $\tilde{E}$.

(3) If $\lambda \geq 0$ is a finitely additive measure on R such that $\mu \ll_p \lambda$ for each continuous semi-norm p, then $T\lambda \geq 0$ is a countably additive measure on R_s such that $T\mu \ll_p T\lambda$ for each continuous semi-norm p.

Proof (1) Let $r, s \in R_s$, $r \cap s = \phi$. Then $\tau^{-1}(r) \cap \tau^{-1}(s) = \tau^{-1}(r \cap s) = \phi$ implies

$$T\mu(r \cup s) = \mu(\tau^{-1}(r \cup s)) = \mu(\tau^{-1}(r) \cup \tau^{-1}(s))$$

$$= \mu(\tau^{-1}(r)) + \mu(\tau^{-1}(s))$$

$$= T\mu(r) + T\mu(s).$$

Now suppose $(r_i) \subseteq R_s$ is disjoint and $r = \cup_i r_i \in R_s$. Then r is compact and covered by (r_i) so that for some $N \in \omega$,

$$r \subseteq \bigcup_{i=1}^{N} r_i.$$

In fact, since $r = \bigcup_i r_i$, we have

$$r = \bigcup_{i=1}^{N} r_i,$$

and hence $r_i = \emptyset$, $i > N$. Then

$$T\mu(\bigcup_i r_i) = T\mu(\bigcup_{i=1}^{N} r_i)$$

$$= \sum_{i=1}^{N} T\mu(r_i) = \sum_i T\mu(r_i),$$

so that $T\mu$ is countably additive.

(2) is obvious from the definition of $T\mu$.

(3) That $T\lambda \geq 0$ is obvious; $T\lambda$ is countably additive by (1). Let U be a neighborhood of 0 in E. Then there is an $\epsilon > 0$ such that $\mu(r) \in U$ for $r \in R$ whenever $\lambda(r) < \epsilon$. But if $T\lambda(r) < \epsilon$ for some $r \in R_s$, then $\lambda(\tau^{-1}(r)) = T\lambda(r) < \epsilon$ also, so that

$$T\mu(r) = \mu(\tau^{-1}(r)) \in U. \quad \square$$

1.10 Definition. Let $\mu: R \to E$ be a measure. Then

(1) $\sigma(R) = \{r \subseteq \Omega: r = \bigcup_i r_i$ for some sequence $(r_i) \subseteq R\}$;

(2) $\rho(R) = \{r \subseteq \Omega: r \subseteq r'$ for some $r' \in \sigma(R)\}$;

(3)a) $D(r,R) = \{r' \in R: r' \subseteq r\}$, for $r \subseteq \Omega$;

b) $\sigma(r,R) = \{r' \in \sigma(R): r' \supseteq r\}$, for $r \subseteq \Omega$;

(4)a) $g(r) = \lim\{\mu(r'): r' \in D(r,R)\}$, $r \subseteq \Omega$,

where $D(r,R)$ is directed by $r' \leq r'' \equiv r' \subseteq r''$

(assuming the limit exists).

b) $\bar{\mu}(r) = \lim\{g(r'): r' \in D_\sigma(r,R)\}$, $r \subseteq \Omega$,

where $\sigma(r,R)$ is directed by $r' \leq r'' \equiv r' \supseteq r''$

(assuming the limit exists).

(5) $M_\mu = \{r \in R:$ for all $s \in R$, $\mu(s) = \mu(s \cap r) + \mu(s-r)\}$.

The next two theorems are due to Sion [15]. Since we need to examine his proofs in detail in Chapter 2, we reproduce them here with a few small changes.

1.11 Theorem. (1) Let $\mu: P(\Omega) \to E$ be such that $\mu(\phi) = 0$. Then M_μ is an algebra and μ is finitely additive on M_μ.

(2) Let $f: D \to E$, where D is a directed set. If for each increasing sequence S in D, $f/_S$ is a Cauchy sequence in E, then f is a Cauchy net in E.

Proof (1) Since $\mu(\phi) = 0$, it is clear that $\Omega \in M_\mu$. By symmetry, if $r \in M_\mu$, $\Omega - r \in M_\mu$ also. Thus if $r, s \in M_\mu$ implies $r \cap s \in M_\mu$, we have that M_μ is an algebra

But for $t \in P(\Omega)$,

$$\mu(t) = \mu(t\cap r) + \mu(t-r)$$

$$= \mu(t\cap r\cap s) + \mu((t\cap r)-s) + \mu(t-r)$$

$$= \mu(t\cap(r\cap s)) + \mu((t-(r\cap s))\cap r) + \mu((t-(r\cap s))-r)$$

$$= \mu(t\cap(r\cap s)) + \mu(t-(r\cap s)).$$

If also $r\cap s = \phi$, then

$$\mu(r\cup s) = \mu((r\cup s)\cap r) + \mu((r\cup s)-r)$$

$$= \mu(r) + \mu(s),$$

so that μ is finitely additive on M_μ.

(2) Suppose f is not Cauchy in E. Then there is some balanced neighborhood of 0 , U, such that, for every $d\in D$, there exists $d'\in D$ with $d' \geq d$ and

$$f(d') - f(d) \notin U.$$

If this were not true, then for $d', d'' \geq d$, $d', d''\in D$,

$$f(d') - f(d'') = f(d') - f(d) - (f(d'') - f(d))$$

$$\in 2U,$$

so that f would be Cauchy.

Choose $d_1\in D$ and pick $d_2\in D$ such that

$$f(d_2) - f(d_1) \notin U.$$

If we have an increasing sequence $(d_i)_{i=1}^{n} \subseteq D$ such that

$$f(d_{i+1}) - f(d_i) \notin U, \qquad 1 \leq i \leq n-1,$$

we can find $d_{n+1} \geq d_n$ using $d = d_n$ in the statement shown above. Thus we have an increasing sequence $(d_i)_i \subseteq D$ which is not Cauchy. □

1.12 Theorem. Suppose the following conditions are satisfied:

(1) $\mu: R \to E$ is countably additive;

(2) $(r_i) \subseteq R$ monotone implies $\lim_i \mu(r_i) \in \widetilde{E}$;

(3) $\Omega \in \sum(R)$.

Then μ has a countably additive extension $\bar{\mu}$ from $\sum(R)$ into $\widetilde{E}$.

Proof We shall break the proof into a sequence of lemmas.

Lemma (a) g is defined on $P(\Omega)$, the set of all subsets of Ω, and $g/_R = \mu$.

Proof For $r \in P(\Omega)$, $\mu/D(r,R)$ forms a Cauchy net, by Theorem 1.11(2) and hypothesis (2) above. That $g/_R = \mu$ is obvious since $r \in R$ implies $r \in D(r,R)$.

Lemma (b) Let $r_n \in R$, $n \in \omega$, $r_n \uparrow_n r$. Then

$$g(r) = \lim_n \mu(r_n).$$

Proof Let U be a balanced, closed neighborhood of 0. Then there exists an $s \in R$ such that $s \subseteq r$ and if $s' \in R$ such that $s' \subseteq r-s$, then $\mu(s') \in U$. Then for $n \in \omega$,

$$\mu(r_n) - \mu(s \cap r_n) = \mu(r_n - s) \in U,$$

while

$$\mu(s) - g(r) = \lim_{t \in D(r,R)} (\mu(s) - \mu(t))$$

$$= \lim_{t \in D(r,R)} -\mu(t-s) \in U.$$

Thus

$$\lim_n \mu(r_n) - g(r) = \lim_n (\mu(r_n) - g(r))$$

$$= \lim_n (\mu(r_n) - \mu(s \cap r_n) + \mu(s \cap r_n) - g(r))$$

$$= \lim_n (\mu(r_n) - \mu(s \cap r_n)) + \mu(s) - g(r)$$

$$\in U+U = 2U.$$

Lemma (c) g is finitely additive on $\sigma(R)$.

Proof Let $(r_{ij})_{j \in \omega} \subseteq R$, $i = 1, 2$. Then if $r_i = \cup_j r_{ij}$, $i = 1, 2$, we have by Lemma (b), if $r_1 \cap r_2 = \phi$,

$$g(r_1 \cup r_2) = \lim_n \mu((\cup_{j=1}^n r_{1j}) \cup (\cup_{j=1}^n r_{2j}))$$

$$= \lim_n (\mu(\cup_{j=1}^n r_{1j}) + \mu(\cup_{j=1}^n r_{2j}))$$

$$= g(r_1) + g(r_2).$$

Lemma (d) Let $r_n \in \sigma(R)$, $n \in \omega$, $r_n \uparrow r$. Then

$$g(r) = \lim_n g(r_n).$$

Proof Choose $s_n \in R$, $n \in \omega$, such that $s_n \uparrow r$ and $s_n \subseteq r_n$, $n \in \omega$. Let U be a neighborhood of 0. Then for each n there exists by recursion a $t_n \in R$ such that

$$s_n \subseteq t_n \subseteq r_n, \; t_n \subseteq t_{n+1}, \; \mu(t_n) - g(r_n) \in U.$$

Since $r = \bigcup_n t_n$, Lemma (b) says there exists an $N \in \omega$ such that for $n \geq N$, $g(r) - \mu(t_n) \in U$. But then

$$g(r) - g(r_n) = g(r) - \mu(t_n) + \mu(t_n) - g(r_n)$$

$$\in U + U = 2U.$$

Lemma (e) Let $(r_n)_n \subseteq P(\Omega)$, $r_n \uparrow$. Then $\lim_n g(r_n)$ exists in $\tilde{E}$.

Proof Let U, V_n, $n \in \omega$ be neighborhoods of 0 such that $\sum_{i=0}^{n} V_i \subseteq U$, $n \in \omega$. Choose $s_n \in R$ so that $s_n \subseteq r_n$ and $\mu(s) \in V_n$ for $s \subseteq r_n - s_n$. Then $g(r_n) - \mu(s_n) \in V_n$ as in Lemma (b). Let $t_{-1} = \Omega$, and for $n \in \omega$, $t_n = \bigcap_{i=1}^{n} s_i$. Then $t_n \in R$, $t_n \downarrow$ and since $(s_n \cap t_{i-1}) - s_i \subseteq r_i - s_i$, for $1 \leq i \leq n$, we have

$$\mu(s_n) - \mu(t_n) = \mu(s_n - t_n) = \sum_{i=0}^{n} \mu(s_n \cap t_{i-1} - s_i)$$

$$\in \sum_{i=0}^{n} V_i \subseteq U.$$

By hypothesis (2) of the theorem $\lim_n \mu(t_n)$ exists since $t_n \downarrow^n$. Thus there is a $N \epsilon \omega$, such that $n \geq N$ implies $\mu(t_n) - \mu(t_N) \epsilon U$. Therefore, for $n \geq N$,

$$g(r_N) - g(r_n) = g(r_N) - \mu(s_N) + \mu(s_N) - \mu(t_N)$$

$$+ \mu(t_N) - \mu(t_n) + \mu(t_n) - \mu(s_n) + \mu(s_n) - g(r_n)$$

$$\epsilon V_N + U + U + U + V_n \subseteq 4U.$$

Lemma (f) $\bar{\mu}$ is defined on $\mathcal{P}(\Omega)$, and $\bar{\mu}/\sigma(R) = g/_\sigma(R)$.

Proof If $r \in \mathcal{P}(\Omega)$, then by Theorem 1.11 (2) and Lemma (e), $g/\sigma(r,R)$ is a Cauchy net in E, so that $\bar{\mu}(r)$ is defined in $\tilde{E}$. That $\bar{\mu}/\sigma(R) = g/\sigma(R)$ is obvious since $r \in \sigma(r,R)$ when $r \in \sigma(R)$.

Lemma (g) If $r \in \mathcal{P}(\Omega)$ and U is a closed, balanced neighborhood of 0, then there exists an $r' \epsilon \sigma(R)$ such that $r \subseteq r'$, $g(r') - \bar{\mu}(r) \epsilon U$, and for every $t \in M_{\bar{\mu}}$ with $t \subseteq r'-r$, $\bar{\mu}(t) \epsilon U$.

Proof Choose $r' \epsilon \sigma(R)$ such that $r \subseteq r'$ and, for every s in $\sigma(R)$ with $r \subseteq s \subseteq r'$, $g(s) - \bar{\mu}(r) \epsilon U$. Then if $t \epsilon M_{\bar{\mu}}$ with $t \subseteq r'-r$, $\bar{\mu}(r \cup t) = \bar{\mu}(r) + \bar{\mu}(t)$, and for every $s \epsilon \sigma(R)$ with $r \cup t \subseteq s \subseteq r'$, $g(s) - \bar{\mu}(r) \epsilon U$. Thus $\bar{\mu}(r \cup t) - \bar{\mu}(r) \epsilon U$ by definition of $\bar{\mu}$, while $\bar{\mu}(t) = \bar{\mu}(r \cup t) - \bar{\mu}(r) \epsilon U$ also.

Lemma (h) Let $a,b \in \sigma(R)$ and U a balanced, closed neighborhood of 0. If $\bar{\mu}(t) \in U$ whenever $t \subseteq a$ or $t \subseteq b$, then for every t such that $t \subseteq a \cup b$, $\mu(t) \in 3U$.

Proof Let $(a_n),(b_n)$ be increasing sequences in R with $a = \bigcup_n a_n$, $b = \bigcup_n b_n$. Then for any $c \in H$ with $c \subseteq a \cup b$, $\mu(c) = \lim_n \mu(c \cap (a_n \cup b_n))$. But $n \in \omega$ implies

$$\mu(c \cap (a_n \cup b_n)) = \mu(c \cap a_n) + \mu(c \cap b_n - a_n) \in 2U.$$

Thus $\mu(c) \in$ closure $(2U) \subseteq 3U$. But then for $s \in \sigma(R)$, $s \subseteq a \cup b$, $g(s) \in 3U$. Therefore if $t \subseteq a \cup b$, then $\bar{\mu}(t) \in 3U$.

Lemma (i) $R \subseteq M_{\bar{\mu}}$.

Proof Let $r \in R$ and $t \subseteq s \in \mathcal{P}(\Omega)$. Then given a balanced, closed neighborhood of 0, U, there is an $a \in \sigma(R)$ so that $t \cap r \subseteq a \subseteq r$ and, for $a' \in \sigma(R)$ with $t \cap r \subseteq a' \subseteq a$, $g(a') - \bar{\mu}(t \cap r) \in U$. Also choose $b \in \sigma(R)$ so that $t - r \subseteq b \subseteq s - r$, and for every $b' \in \sigma(R)$ with $t - r \subseteq b' \subseteq b$, $g(b') - \bar{\mu}(t-r) \in U$. Then for $c \in \sigma(R)$ with $t \subseteq c \subseteq a \cup b$, we have

$$t \cap r \subseteq c \cap r \subseteq a, \quad t-r \subseteq c-r \subseteq b,$$

while by Lemma (c),

$$g(c) = g(c \cup r) + g(c-r).$$

Thus

$$g(c) - (\bar{\mu}(t\cap r) + \bar{\mu}(t-r)) = g(c\cap r) - \bar{\mu}(t\cap r) + g(c-r) - \mu(t-r)$$

$$\in 2U,$$

so that $r \in M_{\bar{\mu}}$ since $\bar{\mu}(t) = \bar{\mu}(t\cap r) + \bar{\mu}(t-r)$.

Lemma (j) $\sigma(R) \subseteq M_{\bar{\mu}}$.

Proof Let $a \in \sigma(R)$ and $t \subseteq s \in \mathcal{P}(\Omega)$. Given U a balanced, closed neighborhood of 0, choose $r_1 \in R$ so that $r_1 \subseteq a$ and, for every $r' \in R$ with $r' \subseteq a-r_1$, $\mu(r') \in U$. Thus if $t' \subseteq a-r_1$, then $g(t') \in U$ and, since $a-r_1 \in \sigma(R)$, $\bar{\mu}(t') \in U$. There exists a $b \in \sigma(R)$ such that $t-a \subseteq b \subseteq s-r_1$ and $g(b) - \bar{\mu}(t-a) \in U$. Then, as in the case of a, there is an $r_2 \in R$ so that $r_2 \subseteq b$ and $\bar{\mu}(t') \in U$ when $t' \subseteq b-r_2$. Then by Lemma (i),

$$\bar{\mu}(t) = \bar{\mu}(t\cap(r_1\cup r_2)) + \bar{\mu}(t-(r_1\cup r_2))$$

$$= \bar{\mu}(t\cap r_1) + \bar{\mu}(t\cap r_2) + \bar{\mu}(t-(r_1\cup r_2)),$$

since $r_1\cap r_2 = \emptyset$. Since $t \subseteq a\cup b$, we have $t-(r_1\cup r_2) \subseteq (a-r_1)\cup(b-r_2)$, so that by Lemma (h), $\mu(t-(r_1\cup r_2)) \in 3U$, and

$$\bar{\mu}(t) - (\bar{\mu}(t\cap r_1) + \bar{\mu}(t\cap r_2)) \in 3U.$$

By Lemma (i), since $t\cap a-r_1 \subseteq a-r$, and $(t-a)-r_2 \subseteq b-r_2$,

$$\bar{\mu}(t\cap a) - \bar{\mu}(t\cap r_1) = \bar{\mu}(t\cap a-r_1) \in U,$$

$$\bar{\mu}(t-a) - \bar{\mu}(t\cap r_2) = \bar{\mu}((t-a)-r_2) \in U.$$

Thus $\bar{\mu}(t) - (\bar{\mu}(t\cap a) + \bar{\mu}(t-a)) \in 5U$.

Lemma (k) Let $a_n \subseteq s \in \mathcal{P}(\Omega)$, $n\in\omega$, so that $a_n \uparrow a$. Then $\bar{\mu}(a) = \lim_n \bar{\mu}(a_n)$.

Proof Let $U,(V_i)_i$ be balanced, closed neighborhoods so that $\sum_{i=0}^{n} V_i \subseteq U$, $n\in\omega$. By Lemma (g), choose $r_n \in \sigma(R)$, $n\in\omega$, so that $a_n \subseteq r_n$, $g(r_n) - \bar{\mu}(a_n) \in V_n$, and for $t\in M_{\bar{\mu}}$ with $t \subseteq r_n - a_n$, $\bar{\mu}(t)\in V_n$. Let $s_{-1} = \phi$, and $s_n = \bigcup_{i=1}^{n} r_i$, $n\in\omega$. Then $s_n - r_n = \bigcup_{i=0}^{n}(r_i-s_{i-1}-r_n)$, $n\in\omega$. Let $c_i = r_i-s_{i-1}-r_n$, $0 \le i \le n$. Then $(c_i)_{i=0}^{n}$ is disjoint; also $c_i \in M_{\bar{\mu}}$ by Theorem 1.11 (1) and Lemma (j). Since $c_i \subseteq r_i-a_n \subseteq r_i-a_i$, $t \in M_{\bar{\mu}}$ implies $\bar{\mu}(t\cap c_i) \in V_i$. Thus since $\bar{\mu}$ is finitely additive on $M_{\bar{\mu}}$, for any $t\in M_{\bar{\mu}}$ we have

$$\bar{\mu}(t\cap(s_n-r_n)) = \sum_{i=0}^{n} \bar{\mu}(t\cap c_i)$$

$$\in \sum_{i=0}^{n} V_i \subseteq U.$$

Let $a' = \cup_i r_i = \cup_i s_i$, so that $a \subseteq a' \in \sigma(R)$, and choose $b \in \sigma(R)$ with $a \subseteq b \subseteq a'$ and $g(b) - \bar{\mu}(a) \in N$. Then for all $n \in \omega$,

$$g(b \cap s_n) - g(b \cap r_n) = \bar{\mu}(b \cap (s_n - r_n)) \in U.$$

Since $a_n \subseteq b \cap r_n \in \sigma(R)$, $g(b \cap r_n) - \bar{\mu}(a_n) \in V_n$ by the choice of a_n. Hence $g(b \cap s_n) - \bar{\mu}(a_n) \in U + V_n \subseteq 2U$. Since by Lemma (f), $\lim_n g(b \cap s_n) = g(b)$, there is an $N \in \omega$ so that, for $n \geq N$, $g(b) - g(b \cap s_n) \in U$. But then

$$\begin{aligned} \bar{\mu}(a) - \bar{\mu}(a_n) &= (\bar{\mu}(a) - g(b)) + (g(b) - g(b \cap s_N)) \\ &\quad + (g(b \cap s_N) - \bar{\mu}(a_n)) \\ &\in U + U + 2U = 4U. \end{aligned}$$

Lemma (l) Let $(a_n)_n \subseteq M_{\bar{\mu}}$, $a_n \uparrow a$. Then $a \in M_{\bar{\mu}}$.

Proof Let $t \subseteq s \in \mathcal{P}(\Omega)$. Define $r_n = a_n \cup (s - a)$. Then $r_n \in \mathcal{P}(\Omega)$, $n \in \omega$, $r_n \uparrow r$, and $t = \cup_n (t \cap r_n)$, so by Lemma (k)

$$\begin{aligned} \bar{\mu}(t) &= \lim_n \bar{\mu}(t \cap r_n) \\ &= \lim_n (\bar{\mu}(t \cap a_n) + \bar{\mu}(t \cap r_n - a_n)) \\ &= \lim_n (\bar{\mu}(t \cap a_n) + \bar{\mu}(t - a)) \\ &= \bar{\mu}(t \cap a) + \bar{\mu}(t - a). \end{aligned}$$

Proof of the theorem By Theorem 1.11 (1), since $\bar{\mu}$ is defined on $\mathcal{P}(\Omega)$, $M_{\bar{\mu}}$ is an algebra, and in fact by Lemma (1), M_{μ} is a σ-algebra. By Lemma (k) $\bar{\mu}$ is countably additive on $M_{\bar{\mu}}$. Since $\bar{\mu}/_R = g/_R = \mu$, $\bar{\mu}$ is the desired extension. □

The following theorem, which we state without proof (See Dunford and Schwartz [6]), is well known and is listed for completeness. The set $ca(\Omega,\sum)$ is the space of countably additive, scalar-valued measures μ on a σ-algebra $\sum$, with norm given by

$$||\mu|| = \nu(\mu)(\Omega).$$

A set $K \subseteq E$ is weakly sequentially compact if every sequence $(k_i)_i \subseteq K$ has a subsequence $(k_{ij})_j$ which converges in E in the weak topology. We shall write $k_i \xrightarrow{w} k$ if a sequence $(k_i)_i$ converges weakly to k.

1.13 Theorem. A set $(\mu_\alpha)_\alpha \subseteq ca(\Omega,\sum)$ is weakly sequentially compact if and only if (μ_α) is uniformly bounded and uniformly countably additive.

We recall that (μ_α) is uniformly countably additive when $r_i \downarrow \phi$ implies $\lim_i \mu_\alpha(r_i) = 0$ uniformly in $\alpha \in A$. (See Dunford and Schwartz [6].)

One of our major interests in the next chapter is that of generalizing the Bartle-Dunford-Schwartz Theorem (Bartle, Dunford, and Schwartz [1]), our Theorem 1.15. The previous proofs (Bartle, Dunford, and Schwartz [1] and Gould [7]) of Theorem 1.15 reduce it, as we do, to the theorem proved in 1.14. Our proof of Theorem 1.14 uses no machinery outside of some elementary properties of the variation of a countably additive, scalar-valued measure, and it is both shorter and more elementary than the earlier proofs.

1.14 Theorem. Let $(\mu_\alpha)_\alpha \subseteq ca(\Omega,\sum)$. If (μ_α) is uniformly countably additive, then there is a countably additive $\lambda \geq 0$ on $\sum$ such that

$$\mu_\alpha << \lambda, \qquad \text{uniformly in } \alpha.$$

Proof We shall first prove that for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ and a finite set $(\mu_i)_{i=1}^n \subseteq (\mu_\alpha)$ such that

$$\nu(\mu_i)(s) < \delta(\epsilon), \qquad 1 \leq i \leq n,$$

implies

$$|\mu_\alpha(s)| < \epsilon, \qquad \alpha \in A,$$

where s is any element of $\sum$. Suppose we deny this for some $\epsilon > 0$. Choose $\mu_1 \in (\mu_\alpha)$. There exists a $\mu_2 \in (\mu_\alpha)$ and

an $s_1 \in \sum$ such that

$$\nu(\mu_1)(s_1) < \frac{\epsilon}{2^2}, \quad |\mu_2(s_1)| \geq \epsilon.$$

If we have a sequence $(s_i)_{i=1}^{k} \subseteq \sum$ and $(\mu_i)_{i=1}^{k+1}$ such that

$$(1) \quad \nu(\mu_i)(s_j) < \frac{\epsilon}{2^{j+1}}, \qquad 1 \leq i \leq j \leq k,$$

$$(2) \quad |\mu_{j+1}(s_j)| \geq \epsilon, \qquad 1 \leq j \leq k,$$

then for the sequence $(\mu_i)_{i=1}^{k+1} \subseteq (\mu_\alpha)$ we can find a $\mu_{k+2} \in (\mu_\alpha)$ and $s_{k+1} \in \sum$ such that

$$\nu(\mu_i)(s_{k+1}) < \frac{\epsilon}{2^{k+2}}, \qquad 1 \leq i \leq k,$$

while

$$|\mu_{k+2}(s_{k+1})| \geq \epsilon.$$

Thus (1) and (2) above hold for $j \geq i \geq 1$ and $j \geq 1$ respectively. Define $t_n = \bigcup_{i \geq n} s_i$. Then for $i \leq n$

$$\nu(\mu_i)(t_n) \leq \sum_{j \geq n} \nu(\mu_i)(s_j)$$

$$\leq \sum_{j \geq n} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2^n},$$

while

$$|\mu_{n+1}(t_n)| \geq |\mu_{n+1}(s_n)| - |\mu_{n+1}(t_n - s_n)|$$

$$\geq \epsilon - \nu(\mu_{n+1})(t_n - s_n)$$

$$\geq \epsilon - \nu(\mu_{n+1})(t_{n+1})$$

$$\geq \epsilon/2 .$$

If we define $t = \cap_n t_n$, then $\mu_i(t) = 0$, $i \geq 1$, so that we have

$$t_n - t \downarrow^n \phi$$

while

$$|\mu_{n+1}(t_n - t)| \geq \epsilon/2, \qquad n \geq 1,$$

contradicting the uniform countable additivity of (μ_α). For convenience, let $\delta_n = \delta(2^{-n})$ and

$$\lambda_n = \sum_{i=1}^{m} \nu(u_{ni}),$$

where $(\mu_{ni})_{i=1}^{m}$ is the set found above corresponding to $\epsilon = 2^{-n}$. Then defining for $s \in \sum$,

$$\lambda(s) = \sum_i \frac{1}{2^i} \frac{\lambda_i(s)}{1 + \lambda_i(\Omega)} ,$$

we have a countably additive measure $\lambda \geq 0$ on $\sum$. Also,

if $s \in \sum$, then

$$\lambda(s) < \delta_n \cdot 2^{-n}/(1 + \lambda_i(\Omega))$$

implies

$$\nu(\mu_{ni})(s) < \delta_n, \qquad 1 \le i \le n,$$

and so that

$$|\mu_\alpha(s)| < \frac{1}{2^n}, \qquad \alpha \in A.$$

Thus λ is the desired measure. □

1.15 Theorem. If $\mu: \sum \to E$ is a countably additive measure, then there exists, for each continuous semi-norm p, a countably additive measure $\lambda_p \ge 0$ on $\sum$ such that

$$\mu \ll_p \lambda_p.$$

Also, λ_p may be chosen so that for $s \in \sum$,

$$\lambda_p(s) \le \sup\{p(\mu(r)): r \subseteq s, r \in \sum\}.$$

Proof Fix p a continuous semi-norm on E. Let $(\mu_\alpha)_{\alpha \in a}$ be the set $\{x\mu: x \in E', |x| \le p\}$. Then $(\mu_\alpha)_\alpha$ is uniformly countably additive since $s_i \downarrow \phi$ implies for $\epsilon > 0$ there is an $N \in \omega$ such that

$$|\mu_\alpha(s_i)| = |x_\alpha(\mu(s_i))| \le p(\mu(s_i))$$

$$< \epsilon, \qquad i \ge N.$$

Since μ is countably additive, μ is bounded. Thus by the first line in the inequality above, $(\mu_\alpha)_\alpha \subseteq c.a(\Omega,\sum)$. By Theorem 1.14, λ_p exists such that

$$\mu_\alpha << \lambda_p,$$

uniformly in α. Thus for $\epsilon > 0$ there is a $\delta > 0$ such that for all α,

$$|\mu_\alpha(s)| < \epsilon \text{ if } \lambda_p(s) < \delta.$$

But then if $\lambda_p(s) < \delta$,

$$p(\mu(s)) = \sup\{|x\mu(s)|: x\epsilon E', |x| \le p\}$$

$$= \sup\{|\mu_\alpha(s)|: \alpha\epsilon A\}$$

$$\le \epsilon.$$

To see that we may require $\lambda_p(s) \le p(\mu)(s)$, $s\epsilon\sum$, consider the choice of λ_p in Theorem 1.14. For $s\epsilon\sum$,

$$\lambda_p(s) = \sum_i \frac{1}{2^i} \cdot \frac{1}{1 + \lambda_i(\Omega)} \sum_{j=1}^{n} \nu(\mu_{ij})(s)$$

$$\le \sup\{\nu(\mu_{ij})(s): \quad i\epsilon\omega, \ 1 \le j \le n=n(i)\}$$

$$\le \sup\{\nu(\mu_\alpha)(s): \alpha\epsilon A\}$$

$$\leq 4 \sup\{|\mu_\alpha|(s): \alpha\in A\}$$

$$= 4 \sup\{|\mu_\alpha(r)|: r \subseteq s, r\in\textstyle\sum, \alpha\in A\}$$

$$\leq 4 \sup\{p(\mu_\alpha r)): r \subseteq s, r\in\textstyle\sum\}.$$

If we define $\lambda_p' = \frac{1}{4}\lambda_p$, λ_p' has all the desired properties. □

Both Bartle, Dunford and Schwartz [1] and Gould [7] proved Theorem 2.15 for a σ-algebra. Dinculeanu and Kluvanek [5] then extended the theorem to measures on σ-ring by reducing this case to the case of a σ-algebra. With our proof this extra work can be eliminated. In fact if $\lambda_i(\Omega)$ is replaced by a bound B_i for λ_i on $\sum$, the proofs of 2.14 and 2.15 are valid for a σ-ring.

CHAPTER II

The Bartle-Dunford-Schwartz Theorem For Strongly Bounded Measures

When working with finitely additive measures, it is frequently necessary to impose various conditions on the measure or its range space in order to obtain meaningful results. At the same time any such condition should satisfy the following: (1) the condition should not be too restrictive, and (2) it should have a natural form. One such condition is contained in the following definition.

2.1 Definition. A finitely additive measure $\mu:R \to E$ is strongly bounded if, given a disjoint sequence $(r_i) \subseteq R$,

$$\lim_i \mu(r_i) = 0.$$

We note from condition (2) of Theorem 2.2 below that a strongly bounded measure μ is "almost" countably additive. Yet as we shall see later, for many spaces, every bounded measure μ is strongly bounded! Strongly bounded, finitely additive measures were first considered in Rickart [14]. Since every countably additive measure μ on a σ-ring is strongly bounded, it is perhaps understandable that very little work was done using the

condition until the work of Brooks and Jewitt [3], where the condition of strong boundedness plays a fundamental role in the study of finitely additive vector measures.

2.2 Theorem. For a finitely additive measure $\mu: R \to E$, the following are equivalent:

(1) μ is strongly bounded;

(2) given a disjoint sequence $(r_i) \subseteq R$,

$$\sum_i \mu(r_i) \in \widetilde{E};$$

(3) given a monotone increasing sequence $(r_i) \subseteq R$,

$$\lim_i \mu(r_i) \in \widetilde{E};$$

(4) given a disjoint sequence $(r_i) \subseteq R$,

$$\lim_{m,n} p(\mu)(\cup_{i=m}^{n} r_i) = 0;$$

for each continuous semi-norm p.

Proof We shall show $(1 \to 4 \to 3 \to 2 \to 1)$.

$(1 \to 4)$: Deny (4). Then for some $\epsilon > 0$ and some continuous semi-norm p, there exists a double sequence $(m_j, n_j)_j$ such that $m_{j+1} \geq n_{j+1} > m_j$, $j \in \omega$, and $p(\mu)(\cup_{i=n_j}^{m_j} r_i) > \epsilon$, $j \in \omega$. Defining $s_j = \cup_{i=n_j}^{m_j} r_i$, we have

$(s_j)_j \subseteq R$ a disjoint sequence such that $p(\mu)(s_j) > \epsilon$, $j\epsilon\omega$. If we choose $t_j \epsilon R$, $t_j \subseteq s_j$, such that $p(\mu(t_j)) > \epsilon$, we clearly contradict (1) with the sequence (t_j).

(4 → 3): Let (r_i) be a monotone increasing sequence in R. Defining $s_{i+1} = r_{i+1} - r_i$, $i\epsilon\omega$, we have $(s_{i+1})_{i\epsilon\omega} \subseteq R$ disjoint. Then by (4), if $\epsilon > 0$, and p, a continuous semi-norm, are fixed, there exists an $N\epsilon\omega$ such that $p(\mu(\cup_{i=n+1}^{m} s_i)) < \epsilon$ for all $m \geq n \geq N$. Since $\mu(\cup_{i=n+1}^{m} s_i) = \mu(r_m) - \mu(r_n)$, we have (3).

(3 → 2): Let $(r_i) \subseteq R$ be disjoint. Then since $(\cup_{i=1}^{n} r_i)_n$ is a monotone increasing sequence in R,

$$\sum_i \mu(r_i) = \lim_n \sum_{i=1}^{n} \mu(r_i) = \lim_n \mu(\cup_{i=1}^{n} r_i)$$

exists in $\tilde{E}$.

(2 → 1): Since the sum converges in $\tilde{E}$, the terms must converge to 0 in E. □

2.3 Corollary If μ is strongly bounded, then

$$\lim_i \mu(r_i) \epsilon \tilde{E},$$

for any monotone sequence $(r_i) \subseteq R$.

Proof If (r_i) is monotone decreasing in R, then $(r_i - r_{i+1})_i$ is disjoint in R, so that as in (4 → 3) above,

$$p(\mu(r_m) - \mu(r_n)) = p(\sum_{i=m}^{n-1} \mu(r_i - r_{i+1}))_{n,m} \to 0,$$

providing $m \leq n$. □

2.4 Example If E is not restricted, then the only condition on R that will insure that a finitely additive measure $\mu: R \to E$ is strongly bounded, is that R be finite. In fact let R be an infinite set, as well as a ring, and choose $E = \mathcal{L}_\infty(R)$, the Banach space of bounded, measurable functions f on Ω with the supremum norm,

$$||f||_\infty = \sup\{|f(x)|: x\epsilon\Omega\}.$$

We shall show that there exists a disjoint sequence $(r_i)_i$ of non-null sets in R. Given that, define $\mu: R \to \mathcal{L}_\infty(R)$ by

$$\mu(r) = \xi_r, \qquad r\epsilon R.$$

Then $||\mu(r_i)||_\infty = 1$, $i\epsilon\omega$, so that μ is not strongly bounded. To prove our earlier assertion, suppose such a sequence does not exist in R. Then there is a disjoint $(r_{1i})_{i=1}^{n_1} \subseteq R$ of non-null sets such that $r\epsilon R$ implies $r \subseteq \cup_{i=1}^{n} r_{1i}$. We may assume $n_1 \geq 2$. Since R is infinite at least one of $\{R/r_{1i}\}_{i=1}^{\infty}$ is infinite also. Say R/r_{1i}.

Then we can find a disjoint $(r_{2i})_{i=1}^{n_2} \subseteq R/r_{11}$ of non-null sets such that $r \in R/r_{11}$ implies $r \subseteq \cup_{i=1}^{m} r_{2i}$. Again we can take $n_2 \geq 2$. Continuing by induction we get $(r_{ij})_{j=1}^{n_i}$, $n_i \geq 2$, such that for all i, $\cup_{j=1}^{n_i} r_{ij} \supsetneq \cup_{j=1}^{n_{i+1}} r_{i+1,j}$. Denoting $\cup_{j=1}^{n_i} r_{ij}$ by r_i, $(r_i - r_{i+1})_i$ is a disjoint sequence in R since $r_i \supseteq r_{i+1}$; since $r_i \neq r_{i+1}$ for all i, none of the sets are null.

Having seen that boundedness and strong boundedness are different concepts, the natural question is under what conditions does one of these concepts imply the other? The first result in this direction was proved by Rickart [14]. Because of its importance, we state it as a theorem without proof.

2.5 Theorem. A finitely additive measure $\mu: R \to E$ is bounded when it is strongly bounded.

This theorem is a corollary to Theorem 2.16 below. Rickart [14] also noted that if E is an Euclidian n-space, the two concepts are identical.

2.6 Theorem. Let $\mu: R \to \Re^n$ be a finitely additive measure. Then μ is strongly bounded if and only if μ is bounded.

Proof The necessity is Theorem 2.5. Conversely, if μ is bounded, so is $\pi_i\mu$, $1 \leq i \leq n$, where π_i is the i^{th} projection map. If we prove the sufficiency for real-valued measures, then $\pi_i\mu$ is strongly bounded for $1 \leq i \leq n$, so that μ is also strongly bounded. Thus let $\mu: R \to \Re$; $\mu = \mu^+ - \mu^-$, so that if μ is not strongly bounded, at least one of μ^+, μ^- is also not strongly bounded. We assume μ^+ is not strongly bounded. Then there is some $\epsilon > 0$ and some disjoint sequence $(r_i)_i \subseteq R$ such that $\mu^+(r_i) > \epsilon$, $i\epsilon\omega$. But then if $s_i = \bigcup_{j=1}^{i} r_i$, we have $(s_i)_i \subseteq R$ while

$$\mu^+(s_i) = \sum_{j=1}^{i} \mu^+(r_i) > i\epsilon, \qquad i\epsilon\omega.$$

Thus μ^+ is not bounded, so that by (2) of Corollary 1.4, μ is not bounded. □

2.7 Corollary. If $\mu: R \to \Re$ is a finitely additive measure, then μ is strongly bounded if and only if μ^+ and μ^- are strongly bounded.

Proof By (2) of Corollary 1.4, μ is bounded if and only if μ^+ and μ^- are bounded. □

Among the spaces E in which boundedness and strong boundedness are equivalent is the space $ba(\Omega,A)$, the

space of bounded, finitely additive, scalar-valued measures μ on an algebra A with norm $||\mu|| = \nu(\mu)(\Omega)$, which will be considered in Chapter 3. That the two conditions are equivalent follows immediately from the next theorem, since $ba(\Omega, A)$ is weakly sequentially complete (Dunford and Schwartz, [6]). A space E is weakly sequentially complete if every sequence Cauchy in E with the weak topology converges weakly in E.

2.8 Theorem. Let $\mu: R \to E$ be a finitely additive measure into a weakly sequentially complete space. Then μ is strongly bounded if and only if μ is bounded.

Proof We need only to show that a bounded measure is strongly bounded, since the converse implication is Theorem 2.5. Since μ is bounded, $x\mu$ is also bounded and hence strongly bounded, for $x \in E'$. Thus if $(r_i) \subseteq R$ is monotone, $x\mu(r_i)$ converges, $x \in E$, so that $\mu(r_i)$ converges weakly to some point $e \in E$ since E is weakly sequentially complete. Therefore, if $(s_i) \subseteq R$ is a disjoint sequence,

$$\sum_i \mu(s_i) = \lim_i \mu(\cup_{j=1}^{i} s_j)$$

exists in E given its weak topology. Since this is also true for any subsequence of (s_i), the Orlicz-Pettis

Theorem (McArthur [12]) says that, in particular,

$$\sum_i \mu(s_i)$$

converges in E. Thus μ is strongly bounded by Theorem 2.2(2). □

2.9 Corollary. If $\mu: R \to E$ is a finitely additive measure into a semi-reflexive space, then μ is bounded only when μ is strongly bounded.

Proof Any semi-reflexive space is weakly sequentially complete (Köthe [11]). □

Along with the condition of strong boundedness of a measure μ is the condition of uniform strong boundedness of a set $(\mu_\alpha)_\alpha$ of measures with a common range space. This notion will be used extensively in Chapter 3.

2.10 Definition Let (μ_α) be a set of finitely additive measures on R into E. Then (μ_α) is uniformly strongly bounded if $(r_i)_i$ disjoint in R implies

$$\lim_i \mu_\alpha(r_i) = 0,$$

uniformly in α.

We next establish some equivalent conditions to uniform strong boundedness. Condition (6) of Theorem

2.11 is the form of uniform strong boundedness used by Brooks and Jewitt [3], where it is called "uniform additivity".

2.11 Theorem. For a set (μ_α) of finitely additive measures on R into E, the following are equivalent:

(1) (μ_α) is uniformly strongly bounded;

(2) given (r_i) disjoint in R,

$$\sum_i \mu_\alpha(r_i)$$

converges uniformly in α to a point in $\widetilde{E}$;

(3) if (r_i) is monotone increasing in R, then $\mu_\alpha(r_i)$ converges uniformly in α to a point in $\widetilde{E}$;

(4) given (r_i) disjoint in R,

$$\lim_{m,n} p(\mu_\alpha)(\cup_{i=m}^{n} r_i) = 0,$$

uniformly in α, for each continuous semi-norm p;

(5) for each continuous semi-norm p, if $(x_\beta) \subseteq E'$ and $|x_\beta| \leq p$, $\beta \in \mathfrak{B}$, then $(x_\beta \mu_\alpha)_{\alpha,\beta}$ is uniformly strongly bounded;

(6) given (r_i) disjoint in R,

$$\lim_n \sum_{i \in \Delta \cap [n,\infty)} \mu_\alpha(r_i) = 0,$$

uniformly in $\alpha\in\mathcal{G}$ and in $\Delta \subseteq \omega$.

Proof We show $(1 \to 5 \to 4 \to 3 \to 2 \to 1)$ and $(4 \to 6 \to 1)$.

$(1 \to 5)$ Let $(r_i)_i$ be a disjoint sequence in R. Then if $\epsilon > 0$, there exists an $N\in\omega$ such that $p(\mu_\alpha(r_i)) < \epsilon$ for $i \geq N$ and $\alpha\in\mathcal{G}$. Since $|x_\beta \mu_\alpha(r_i)| \leq p(\mu_\alpha(r_i))$, $(x_\beta \mu_\alpha)$ is uniformly strongly bounded.

$(5 \to 4)$ Deny (4). Then for some $\epsilon > 0$, if $i\in\omega$, there are n_i, $m_i\in\omega$ with $n_i \geq m_i \geq i$ and $\mu_{\alpha_i} \in (\mu_\alpha)$ such that $p(\mu_{\alpha_i})(\bigcup_{j=m_i}^{n_i} r_j) > \epsilon$. Clearly we may assume $m_{i+1} > n_i$. Then defining $s_i = \bigcup_{j=m_i}^{n_i} r_j$, we have $(s_i) \subseteq R$ disjoint such that $p(\mu_{\alpha_i})(s_i) > \epsilon$, $i\in\omega$. Choosing $t_i\in R$, $t_i \subseteq s_i$ such that $p(\mu_{\alpha_i}(t_i)) > \epsilon$, we contradict (5) by choosing $x_i\in E'$ with $|x_i\mu_{\alpha_i}(t_i)| = p(\mu_{\alpha_i}(t_i))$.

$(4 \to 3 \to 2 \to 1)$ These follow as in the proof of Theorem 2.2, allowing μ to run over the μ_α.

$(4 \to 6)$ If p is a continuous semi-norm on E,

$$p\Big(\sum_{i\in\Delta\cap[n,\infty)} \mu_\alpha(r_i)\Big) \leq \sum_{i\geq n} p(\mu_\alpha)(r_i),$$

so that the limit in (6) is uniform in $\Delta \subseteq \omega$.

(6 → 1) Let $\Delta = \omega$. Then since the sum in (6) is uniform in α, $\mu_\alpha(r_i)$ converges to 0 uniformly in α. □

2.12 Corollary. If $\mu_\alpha: R \to \mathbb{R}^n$, $\alpha \in G$, then the following are equivalent.

(1) (μ_α) is uniformly strongly bounded;

(2) given a disjoint $(r_i)_i \subseteq R$,

$$\sum_i |\mu_\alpha(r_i)|$$

converges in $\mathbb{R}$ uniformly in α;

(3) given a disjoint $(r_i)_i \subseteq R$,

$$\lim_n \sum_{i \in \Delta \cap [n,\infty)} |\mu_\alpha(r_i)| = 0$$

uniformly in $\Delta \subseteq \omega$ and in $\alpha \in G$.

Proof Suppose (1) holds. We can assume that μ is real valued since (2) will hold when (2) holds for each $\pi_i\mu$, $1 \le i \le n$, where π_i is the i^{th} projection map. But for $m, n \in \omega$ there is a $\Delta \subseteq [m,n]$ such that

$$\sum_{i=m}^{n} |\mu(r_i)| = \sum_{i \in \Delta} \mu(r_i) - \sum_{i \notin \Delta} \mu(r_i)$$

$$\le 2 \sup\{|\textstyle\sum_{i \in \Delta'} \mu(r_i)| : \Delta' \subseteq [m,n]\},$$

so that (2) holds by (4) of Theorem 2.11. (2 → 3 → 1) is

obvious. □

2.13 Corollary. If $\mu: R \to E$ is a finitely additive measure, then μ is strongly bounded if and only if, for each continuous semi-norm p, if $(x_\alpha) \subseteq E'$ and $|x_\alpha| \leq p$, $\alpha \in G$, then $(x_\alpha \mu)$ is uniformly strongly bounded.

2.14 Corollary. If (μ_α) is as in the above theorem, but E is a Banach space, then the following are equivalent:

(1) (μ_α) is uniformly strongly bounded;

(2) If (r_i) is pair-wise-disjoint in R, then

$$\lim_{m,n} ||\mu_\alpha||(\cup_{i=m}^{n} r_i) = 0,$$

uniformly in α;

(3) if $(x_\beta) \subseteq E'$, $||x_\beta|| \leq 1$, $\beta \in B$, then $(x_\beta \mu_\alpha)_{\alpha,\beta}$ is uniformly strongly bounded.

Proof The proof of Theorem 2.11 holds if the p's are restricted to any set of continuous semi-norms which generates the topology on E. □

Having seen the relations between boundedness and strong boundedness, one might hope for similar relations between the corresponding uniform conditions. The following simple examples show, however, that the conditions are not comparable.

2.15 Example. Let (r_i) be a disjoint sequence of non-null sets, and let R be the ring generated by (r_i). We define $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise.

(1) Let $\mu_i: R \to \mathfrak{R}^+$ be the measure determined by $\mu_i(r_j) = \delta_{ij}$. Then $(\mu_i)_i$ is uniformly bounded by 1, but obviously (μ_i) is not uniformly strongly bounded.

(2) Let $\mu_i: R \to \mathfrak{R}^+$ be the measure determined by $\mu_i(r_j) = i\,\delta_{1j}$. Then (μ_i) is not uniformly bounded, but it is uniformly strongly bounded.

The next theorem shows that it was not an accident in (2) above that the sequence (μ_i) was unbounded on a single $r \in R$. Example (1) above shows that the converse of Theorem 2.16 fails.

2.16 Theorem. Let $\mu_\alpha: R \to E$, $\alpha \in \mathfrak{A}$ be finitely additive measures. If (μ_α) is uniformly strongly bounded and point-wise bounded, then (μ_α) is uniformly bounded.

Proof Let p be a continuous semi-norm on E. For $r \in R$, we define $B(r) = \sup\{p(\mu_\alpha(r)): \alpha \in \mathfrak{A}\}$ and $S(r) = \sup\{B(s): s \in R, s \subseteq r\}$. Since (μ_α) is point-wise

bounded, we have $B(r) < \infty$, $r \in R$, by definition. We first show that (μ_α) is p-bounded on $R_{/r}$, the algebra induced by R on one of its elements r. Suppose not, and let $t_0 = r$. Then $S(t_0) = \infty$, so that there exists $s_1 \in R$, $s_1 \subseteq r$ such that for some $\mu_{\alpha_1} \in (\mu_\alpha)$, $p(\mu_{\alpha_1}(s_1)) > B(t_0) + n_1$, where $n_1 = B(t_0) + 1$. Then $p(\mu_{\alpha_1}(s_1)) > n_1$ obviously, while

$$\begin{aligned} p(\mu_{\alpha_1}(t_0 - s_1)) &\geq p(\mu_{\alpha_1}(s_1)) - p(\mu_{\alpha_1}(t_0)) \\ &\geq p(\mu_{\alpha_1}(s_1)) - B(t_0) \\ &\geq n_1, \end{aligned}$$

also. We choose t_1 to be s_1 or $t_0 - s_1$, so that $S(t_1) = \infty$. Suppose we have a monotone decreasing sequence $(t_i)_{i=0}^m$ so that defining $n_i = B(t_{i-1}) + 1$, $1 \leq i \leq m$, we have for some $(\mu_{\alpha_i})_{i=1}^m \subseteq (\mu_\alpha)$,

$$(1) \quad p(\mu_{\alpha_i}(t_i)) > n_i,$$

and

$$(2) \quad S(t_i) = \infty,$$

for $1 \leq i \leq m$. Setting $n_{m+1} = B(t_m) + 1$, we may choose by (2) above, an $s_{m+1} \in R$, $s_{m+1} \subseteq t_m$, so that

$$p(\mu_{\alpha_{m+1}}(s_{m+1})) > B(t_m) + n_{m+1}.$$

As before, we also have $p(\mu_{\alpha_{m+1}}(t_m - s_{m+1})) > n_{m+1}$. Choose t_{m+1} to be s_{m+1} on $t_m - s_{m+1}$, so that $S(t_{m+1}) = \infty$. Thus we have $(\mu_{\alpha_i})_{i>0} \subseteq (\mu_\alpha)$ and $(t_i)_{i\geq 0} \subseteq R$ such that (1) above holds for $i > 0$ when $n_i = B(t_{i-1}) + 1$. But then

$$\begin{aligned} p(\mu_{\alpha_i}(t_{i-1} - t_i)) &\geq p(\mu_{\alpha_i}(t_i)) - p(\mu_{\alpha_i}(t_{i-1})) \\ &\geq p(\mu_{\alpha_i}(t_i)) - B(t_{i-1}) \\ &\geq 1, \end{aligned}$$

contradicting the uniform strong boundedness of (μ_α).

Now suppose that (μ_α) is not p-bounded on R. Then there exists $(r_i) \subseteq R$ such that $p(\mu_{\alpha_i}(r_i)) \uparrow \infty$, where $(\mu_{\alpha_i}) \subseteq (\mu_\alpha)$. We may also require that, for $i \in \omega$,

$$p(\mu_{\alpha_{i+1}}(r_{i+1})) > 2 \sum_{j=0}^{i} S(r_j).$$

Defining $s_{i+1} = r_{i+1} - (\cup_{j=1}^{i} r_j)$, $i \in \omega$, we see that

$$p(\mu_{\alpha_{i+1}}(s_{i+1})) \geq p(\mu_{\alpha_{i+1}}(r_{i+1})) - p(\mu_{\alpha_{i+1}}(r_{i+1} \cap (\cup_{j=1}^{i} r_j)))$$

$$\geq p(\mu_{\alpha_{i+1}}(r_{i+1})) - \sum_{j=0}^{i} S(r_j)$$

$$\geq \frac{1}{2} p(\mu_{\alpha_{i+1}}(r_{i+1})),$$

so that we again contradict the uniform strong boundedness of the (μ_α), since the $(s_i) \subseteq R$ pair-wise-disjoint.

Since p was an arbitrary continuous semi-norm on E, (μ_α) is bounded. □

2.17 Remark. As we have seen in Example 2.15, Theorem 2.16 is false, in general, if (μ_α) is not required to be point-wise bounded. This is so even if E is restricted to be the scalars. Thus since a set $K \subseteq ba(\Omega,A)$ must be bounded if it is weakly sequentially compact, the sufficiency in Theorem IV 9.12 of Dunford and Schwartz [6] is false without the added hypothesis that K be bounded (or at least point-wise bounded, using Theorem 2.16).

2.18 Theorem. Let (μ_α) be a set of finitely additive measures on R into $\mathbb{R}$. Then (μ_α) is uniformly strongly bounded if and only if (μ_α^+) and (μ_α^-) are uniformly strongly

bounded.

Proof Since $|\mu(r)| \leq \nu(\mu)(r) = \mu^+(r) + \mu^-(r)$, $r \in R$, by Theorem 1.13, the sufficiency is obvious. Suppose (μ_α^+) is not uniformly strongly bounded. Then for some $(r_i) \subseteq R$ pair-wise-disjoint, there exist integers m_i and $\mu_{\alpha_i} \in (\mu_\alpha)$, $i \in \omega$, such that $m_i \uparrow_i \infty$, and for some $\epsilon > 0$,

$$\sum_{m_i \leq j \leq m_{i+1}} \mu_j^+(r_j) > \epsilon.$$

For each $i \in \omega$, if $m_i \leq j < m_{i+1}$ and $\mu_{\alpha_i}^+(r_j) > 0$, choose $s_j \in R$, $s_j \subseteq r_i$ so that $\mu_{\alpha_i}(s_j) > \frac{1}{2}\mu_{\alpha_i}^+(r_i)$; otherwise choose $s_i = \emptyset$. Since (μ_α) is uniformly strongly bounded, there exists an $N \in \omega$ such that $n \geq N$ implies

$$\left|\sum_{j \geq n} \mu_\alpha(s_j)\right| < \epsilon/4,$$

for every $\alpha \in \mathfrak{A}$. But if $m_i \geq N$,

$$\begin{aligned}\left|\sum_{j \geq m_i} \mu_{\alpha_i}(s_j)\right| &\geq \left|\sum_{m_i \leq j \leq m_{i+1}} \mu_{\alpha_i}(s_j)\right| - \left|\sum_{j \geq m_{i+1}} \mu_{\alpha_i}(s_i)\right| \\ &> \frac{1}{2}\sum_{m_i \leq j < m_{i+1}} \mu_{\alpha_i}^+(r_j) - \epsilon/4 \\ &> \epsilon/4,\end{aligned}$$

a contradiction. □

2.19 Example. Let R be the ring of all subsets of $\mathfrak{R}$ of finite Lebesque measure. Then R is obviously a δ-ring. If μ is the Lebesque measure on $\mathfrak{R}$, then $\mu_{/R}$ is a countably additive, non-negative, finite-valued measure on a δ-ring which is not bounded.

Since any vector measure on a σ-ring is bounded, Theorem 2.20 below is a generalization of the Bartle-Dunford-Schwartz Theorem, Theorem 1.15, for measures on a δ-ring. Example 2.19 shows that it is necessary in Theorem 2.20 to assume that each λ_p is bounded since the λ there is absolutely continuous with respect to itself but is not strongly bounded.

2.20 Theorem. If $\mu\colon R \to E$ is a countably additive measure on a δ-ring R, then μ is strongly bounded if and only if, for each continuous semi-norm p, μ has a bounded p-control measure λ_p.

Proof ($\leftarrow$) A basic neighborhood of 0 in E is of the form $\{e \in E\colon p_i(e) \leq 1,\ 1 \leq i \leq n\}$. Let (r_j) be disjoint in R. Then for $1 \leq i \leq n$ there exists an $N_i \in \omega$ such that $j \geq N_i$ implies $\lambda_{p_i}(r_j) < \delta_i(1)$. If $N = \max\{N_i\colon 1 \leq i \leq n\}$, then $j \geq N$ implies $p_i(\mu(r_j)) < 1$,

$1 \leq i \leq n$. Thus $\mu(r_j) \underset{j}{\to} 0$.

($\rightarrow$) We assume $\mu \not\equiv 0$. Then by Zorn's Lemma, there is a set M of pair-wise disjoint elements of R for which $\mu(r) \not\equiv 0$, $r \in M$, and such that if $r' \in R$ is disjoint from each $r \in M$, then $\mu(r') = 0$. Fix p a continuous semi-norm, and define, for $i \in \omega$, $M_i = \{r \in M: p(\mu(r)) > 1/i\}$. Since M is pair-wise-disjoint and μ is strongly bounded, M_i is finite for each $i \in \omega$. Let $M = \cup_i M_i = (r_i)_{i \in \omega}$. Since the Bartle-Dunford-Schwartz Theorem, Theorem 1.15, holds for μ restricted to R/r_i. $i \in \omega$, there exists for each i a p-control λ_i on R/r_i such that $\lambda_i(r_i) \leq B$, where B is a bound for μ on R (Theorem 2.5). Thus defining

$$\lambda(r) = \sum_i \frac{1}{2^i} \lambda_i(r \cap r_i), \qquad r \in R,$$

gives a bounded, non-negative countably additive measure on R. By Theorem 2.2(4), there is an N such that $n \geq m \geq N$ implies

$$p(\mu)(\cup_{i=m}^{n} r_i) < \epsilon,$$

where $\epsilon > 0$ is fixed. Thus if $r \in R$,

$$p(\mu(r\cap U_{i\geq N} r_i)) = \lim_n p(\mu(r\cap U_{i=N}^n r_i))$$

$$\leq \limsup_n p(\mu)(U_{i=N}^n r_i)$$

$$\leq \epsilon.$$

Now, for $1 \leq i \leq N-1$, there exists a δ_i such that

$$r\epsilon R_{/r_i},\ \lambda_i(r) < \delta_i \text{ implies } p(\mu(r)) < \epsilon/N.$$

Set $\delta = \frac{1}{2^N} \min\{\delta_i : 1 \leq i \leq N-1\}$. Then $r\epsilon R$, $\lambda(r) < \delta$ implies

$$p(\mu(r)) = p(\mu(r\cap U_i r_i))$$

$$\leq \sum_{i=1}^{N-1} p(\mu(r\cap r_i)) + p(\mu(r\cap U_{i\geq N} r_i))$$

$$< N \cdot \frac{\epsilon}{N} + \epsilon = 2\epsilon.$$

In fact, $\mu(r - U_{i\geq 1} r_i) = 0$ since $r - U_{i\geq 1} r_i$ is disjoint from each element of $M = (r_i)$, so that $\mu(r) = \mu(r - U_{i\geq 1} r_i)$. Thus we may set $\lambda_p = \lambda$. □

Theorem 2.20 is in fact true under much more general conditions, but before we prove this, we need a few more theorems. The set M_μ was defined in Definition 1.10(5).

Lemma 2.21. Let R be a ring and μ a function on R into E with $\mu(\phi) = 0$. Then M_μ is a subring of R, and μ is finitely additive on M_μ.

Proof We obviously have $\phi \in M_\mu$ from $\mu(\phi) = 0$. We shall first show that M_μ is closed under relative complements. Suppose that $a, b \in M_\mu$ and $t \in R$. By set identities we have $\mu(t \cap (a-b)) + \mu(t-(a-b)) = \mu((t-b) \cap a) + \mu((t-a) \cup (t \cap b \cap a))$. But from $b \in M_\mu$ we also have $\mu((t-a) \cup (t \cap b \cap a)) = \mu((t-a)-b) + \mu((t \cap b \cap a) \cup ((t-a) \cap b)) = \mu((t-b)-a) + \mu(t \cap b)$ (the last equality by set identity). Thus $\mu(t \cap (a-b)) + \mu(t-(a-b)) = \mu((t-b) \cap a) + \mu((t-b)-a) + \mu(t \cap b) = \mu(t-b) + \mu(t \cap b) = \mu' t)$ using $a, b \in M_\mu$. We next show that M_μ is closed under unions, completing our proof since μ is obviously finitely additive on M_μ. By set identities again (with a, b, t as above) $\mu(t \cap (a \cup b)) + \mu(t-(a \cup b)) = \mu((t \cap a) \cup ((t-a) \cap (b-a))) + \mu((t-a)-(b-a))$. But $\mu((t \cap a) \cup ((t-a) \cap (b-a))) = \mu(t \cap a) + \mu((t-a) \cap (b-a))$ since $a \in M_\mu$. Thus $\mu(t \cap (a \cup b)) + \mu(t-(a \cup b)) = \mu(t \cap a) + \mu((t-a) \cap (b-a)) + \mu((t-a) - (b-a)) = \mu(t \cap a) + \mu(t-a) = \mu(t)$, using $b-a, a \in M_\mu$. □

2.22 Theorem. If $\mu: R \to E$ is a countably additive, strongly bounded measure, then μ can be extended to a countably additive measure $\bar{\mu}$ on $\sum$, the σ-ring generated by R, into $\widetilde{E}$.

Proof We shall follow the proof of Theorem 1.12, with some modifications. The proofs of Lemma (a) to Lemma (e) are unchanged. But instead of defining $\bar{\mu}$ on all subsets of Ω, as in Lemma (f), we will define $\bar{\mu}$ only on $\rho(R)$. It is obvious that $\bar{\mu}$ is defined on $\rho(R)$ since Lemma (e) and Theorem 1.11(2) show that for $t \subseteq \Omega$,

$$g/\sigma(t,R)$$

is Cauchy whenever $\sigma(t,R) \neq \phi$. The fact that

$$\bar{\mu}/_{\sigma(R)} = g/_{\sigma(R)}$$

follows easily as before. Since the definition of $M_{\bar{\mu}}$ depends on the domain of $\bar{\mu}$, the $M_{\bar{\mu}}$ we have now is changed, but Lemma 2.21 shows that it is still a ring. If we replace $P(\Omega)$ with $\rho(R)$ in Lemma (f) through Lemma (m), then the theorem follows, since in those proofs we only used the facts that $M_{\bar{\mu}}$ is a ring, that $P(\Omega)$ is closed under finite intersections and countable unions, and that each element of $P(\Omega)$ is a subset of a countable union of

elements of R. □

Although we don't need this next theorem to prove Theorem 2.25, we shall require it in Chapter 3, and this is a convenient place to prove it.

2.23 Theorem. Let μ and μ_α, $\alpha\in\mathfrak{A}$, be countably additive, strongly bounded measures on R into E. Then

(1) if (μ_α) is uniformly bounded, then so is $(\bar{\mu}_\alpha)$;

(2) if μ has a bounded p-control λ_p, for some continuous semi-norm p, then $\bar{\lambda}_p$ is a bounded $\tilde{p}$-control for $\bar{\mu}$;

where $\bar{\mu}$, $\bar{\lambda}_p$ are the extensions of μ, λ_p to $\sum$ into $\tilde{E}$ and $\tilde{p}$ is the continuous extension of p to $\tilde{E}$.

Proof (1) Since a subset of $\tilde{E}$ is bounded if and only if its image under q is bounded, for each continuous semi-norm q on $\tilde{E}$, and since each q is the extension of itself restricted to E, we shall prove that for $s\in\sum$ and p a continuous semi-norm on E,

$$\sup\{\tilde{p}(\bar{\mu}_\alpha(s)) : \alpha\in\mathfrak{A}\} < \infty.$$

Fix $\epsilon > 0$ $s\epsilon\sum$, p a continuous semi-norm on E, and $\alpha\epsilon\mathfrak{A}$. By Lemma (g) of 1.12, there exists an $a\epsilon\sigma(R)$, say $a = \cup_i r_i$, $(r_i) \subseteq R$, such that

$$\bar{\mu}_\alpha(s) - g_\alpha(a) \in V,$$

where $V = \{e\epsilon\tilde{E}: \tilde{p}(e) \leq \epsilon\}$. Since by Lemma (d)

$$g_\alpha(a) = \lim_i \mu_\alpha(\cup_{j=1}^{i} r_i),$$

there also exists an $N_\alpha\epsilon\omega$ such that

$$g_\alpha(a) - \mu_\alpha(\cup_{j=1}^{N_\alpha} r_i) \in V.$$

Thus

$$\tilde{p}(\bar{\mu}_\alpha(s)) \leq \tilde{p}(\bar{\mu}_\alpha(s) - g_\alpha(a)) + \tilde{p}(g_\alpha(a) - \mu_\alpha(\cup_{j=1}^{N_\alpha} r_i)) + p(\mu_\alpha(\cup_{j=1}^{N_\alpha} r_i))$$

$$\leq \epsilon + \epsilon + B = B + 2\epsilon,$$

where B is a uniform bound for $(p(\mu_\alpha(\cdot)))$ on R. Since ϵ is independent of s and of α, we see that

$$\tilde{p}(\bar{\mu}_\alpha(s)) \leq B, \qquad s\epsilon\sum, \ \alpha\epsilon\mathfrak{A}.$$

(2) Set $V = \{e\epsilon\tilde{E}: \tilde{p}(e) \leq \delta\}$. Since

$\mu \ll_p \lambda_p$, there exists an $\epsilon > 0$ such that for $r \in R$,

$$\lambda(r) < \epsilon \text{ implies } \mu(r) \in V.$$

Now let $s \in \sum$ such that $\bar{\lambda}_p(s) < \epsilon/3$. By the definition of $\bar{\mu}$ and $\bar{\lambda}_p$, there exist h_1, $h_2 \in \sigma(R)$ such that $h \subseteq h_1 \cap h_2$ implies

$$|\bar{\lambda}_p(s) - \bar{\lambda}_p(h)| < \frac{\epsilon}{3},\ \bar{\mu}(s) - \bar{\mu}(h) \in V.$$

Since if we set $h = h_1 \cap h_2$, we have $h \in \sigma(R)$, say $h = \cup_i r_i$, $(r_i) \subseteq R$. Then

$$|\bar{\lambda}_p(s) - \bar{\lambda}_p(h)| = |\bar{\lambda}_p(s) - \lim_i \lambda_p(\cup_{j=1}^{i} r_j)|$$

$$= \lim |\bar{\lambda}_p(s) - \lambda_p(\cup_{j=1}^{i} r_j)|$$

and

$$p(\bar{\mu}(s) - \bar{\mu}(h)) = p(\bar{\mu}(s) - \lim_i \mu(\cup_{j=1}^{i} r_j))$$

$$= \lim_i p(\bar{\mu}(s) - \mu(\cup_{j=1}^{i} r_j)),$$

so there exists an $N \in \omega$ such that defining $r = \cup_{j=1}^{N} r_j$, we have

$$|\bar{\lambda}_p(s) - \lambda(r)| < \frac{\epsilon}{3},\ p(\bar{\mu}(s) - \mu(r)) < \delta.$$

But then

$$\lambda_p(r) \leq \bar{\lambda}_p(s) + |\bar{\lambda}_p(s) - \bar{\lambda}(h)| + |\bar{\lambda}(h) - \lambda(r)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

implies $\mu(r) \in V$.

Thus

$$\ddot{\mu}(s) = \bar{\mu}(s) - \ddot{\mu}(h) + \ddot{\mu}(h) - \bar{\mu}(r) + \mu(r)$$

$$\in V + V + V$$

shows that $\tilde{p}(\ddot{\mu}(s)) \leq 3\delta$. Thus $\ddot{\mu} <<_{\tilde{p}} \bar{\lambda}_p$. □

2.24 Theorem. If $\mu: R \to E$ is a countably additive measure, then μ is strongly bounded if and only if, for each continuous semi-norm p, μ has a bounded, countably additive p-control λ_p.

Proof The sufficiency follows as in Theorem 2.20. To prove the necessity, we shall first show that μ has a countably additive, strongly bounded extension $\bar{\mu}$ to the δ-ring D generated by R. For each $r \in R$, denote the extension of μ to the σ-algebra generated by $R/_r$ by $\ddot{\mu}_r$. This extension exists by Theorem 2.22; call the g defined in

the theorem g_r. We define $\bar{\mu}(d) = \bar{\mu}_r(d)$, $d \in D$, where $r \in R$ is chosen so that $d \subseteq r$. Suppose that $r'' \in R$ also, with $d \subseteq r''$. We need to show $\mu_r(d) = \mu_r(d) = \mu_{r''}(d)$. Define $s = r \cap r''$. Since $s \in R/_r \cap R/_{r'}$, if U is a balanced neighborhood of 0, we can find, using the definitions of the functions g and $\bar{\mu}$, $s' \in \sigma(R/_r)$, $s'' \in \sigma(R/_{r''})$, so that $d \subseteq s'$, $s'' \subseteq s$ and if $t' \subseteq s$, $t' \in \sigma(d, R/_r)$, $t'' \subseteq s''$, $t'' \in \sigma(d, R/_{r''})$, then

$$\bar{\mu}_r(d) - g_r(t') \in U, \quad \bar{\mu}_{r''}(d) - g_{r''}(t'') \in U.$$

Since $g_r = g_{r''}$ on elements of $\sigma(R/_r) \cap \sigma(R/_{r''})$, we now have

$$\bar{\mu}_r(d) - \bar{\mu}_{r''}(d) = \bar{\mu}_r(d) - g_r(s' \cap s'') + g_{r''}(s' \cap s'') - \bar{\mu}_{r''}(d)$$

$$\in U - U = 2U,$$

since $s' \cap s'' \in \sigma(d, R/_r) \cap \sigma(d, R/_{r''})$. Thus we see that $\bar{\mu}_r(d) = \bar{\mu}_{r''}(d)$, so that $\bar{\mu}$ is well defined. Moreover, $\bar{\mu}$ is countably additive on D. Suppose $(d_i) \subseteq D$ is pair-wise-disjoint and $d = \bigcup_i d_i \in D$. If we choose $r \in R$ such that $d \subseteq r$, we have

$$\bar{\mu}(d) = \bar{\mu}_r(d) = \sum_i \bar{\mu}_r(d_i) = \sum_i \bar{\mu}(d_i).$$

Now if we show $\bar{\mu}$ to be strongly bounded on D, we can apply Theorem 2.20 to find a p-control λ_p for $\bar{\mu}$ on D, in which case the restriction of λ_p to R will obviously be a p-control for μ. Fix $(d_i) \subseteq D$ disjoint, and U a closed, balanced, convex neighborhood of 0 such that, for each $i\epsilon\omega$,

$$\bar{\mu}(d_i) \notin U.$$

By the continuity of addition at 0, there exists a sequence (V_i) of neighborhoods of 0 such that for all i

$$\sum_{j=1}^{i} V_j \subseteq \frac{1}{4} U.$$

By Lemma (g) of Theorem 1.12, applied for each $i\epsilon\omega$, there exists $(r_{ij})_j \subseteq R$ such that for some $r_i \epsilon R$,

$$r_i \supseteq \bigcup_j r_{ij} = s_i \supseteq d_i,$$

while $$\bar{\mu}(d_i) - \bar{\mu}(s_i) \in V_i,$$

and $$\bar{\mu}(d' \cap s_i - d_i) \in V_i, \qquad d' \epsilon D.$$

Thus for each $i\epsilon\omega$,

$$\bar{\mu}(s_i) \notin \frac{1}{2} U,$$

since otherwise,

$$\bar{\mu}(d_i) = \bar{\mu}(d_i) - \bar{\mu}(s_i) + \bar{\mu}(s_i)$$

$$\in V_i + \frac{1}{2} U \subseteq \frac{1}{4} U + \frac{1}{2} U \subseteq U.$$

Now by Lemma (f) of 1.12,

$$\bar{\mu}(s_i) = g_{T_i}(s_i) = \lim_j \bar{\mu}_{T_i}(\cup_{k=1}^{j} r_{i_k})$$

$$= \lim_j \mu(\cup_{k=1}^{j} r_{ik}),$$

so for each i there exists an n_i such that

$$\mu(\cup_{k=1}^{n_i} r_{ik}) \notin \frac{1}{2} U,$$

since $\frac{1}{2} U$ is closed. If we define $a_i = \cup_{j=1}^{n_i} r_{ij}$, then $(a_i) \subseteq R$ is pair-wise-disjoint, $a_i \subseteq s_i$, $i\in\omega$, while

$$\mu(a_i) \notin \frac{1}{2} U, \ \bar{\mu}(a_i \cap d' - d_i) \in V_i, \qquad i\in\omega, \ d'\in D.$$

We shall show that $h_i = a_i - \cup_{j=1}^{i-1} a_j$, $i\in\omega$, gives a pair-wise-disjoint sequence in R whose images under μ are bounded away from 0. It is enough to show that for each i,

$$\mu(a_i \cap (\cup_{j=1}^{i-1} a_j)) \in \sum_{j=1}^{i} V_j,$$

since then

$$\mu(a_i) = \mu(h_i) + \mu(a_i \cap (\cup_{j=1}^{i-1} a_j))$$

$$\in \mu(h_i) + \sum_{j=1}^{i} V_j$$

$$\subset \mu(h_i) + \frac{1}{4} U,$$

implies that

$$\mu(h_i) \notin \frac{1}{4} U, \qquad i \in \omega,$$

contradicting the strong boundedness of μ on R. But

$$\mu(a_i \cap (\cup_{j=1}^{i-1} a_j)) = \bar{\mu}(d_i \cap a_i \cap (\cup_{j=1}^{i-1} a_j)) + \bar{\mu}((a_i \cap (\cup_{j=1}^{i-1} a_j)) - d_i)$$

$$\in \bar{\mu}(d_i \cap a_i \cap (\cup_{j=1}^{i-1} a_j)) + V_i,$$

while

$$\bar{\mu}(d_i \cap a_i \cap (\cup_{j=1}^{i-1} a_j)) = \sum_{j=1}^{i-1} \bar{\mu}((d_i \cap a_i \cap (a_j - \cup_{k=1}^{j-1} a_k)) - d_j)$$

$$\in \sum_{j=1}^{i-1} V_j,$$

since (d_i) is pair-wise-disjoint. □

The reader may have noticed that we only used the weaker form of Theorem 2.22, that is, the case when $\sum(R)$ is an algebra. This was done since Theorem 1.12 is the only version available in the literature. In fact, however, Theorem 2.24 is a corollary of Theorem 2.22: Given μ

strongly bounded, $\bar{\mu}$ on $\sum = \sum(R)$ is countably additive, so by Theorem 2.20 $\bar{\mu}$ has a p-control $\bar{\lambda}_p$; thus $\lambda_p = \bar{\lambda}_{p/R}$ is the desired p-control for μ on R.

The results so far have been for countably additive measures. Using the powerful representation of R as a Stone ring R_s, however, we can prove our most general result on control measures for finitely additive measures. As mentioned in the introduction, this result was shown by Brooks [2], although we arrived at it independently and with different techniques.

2.25 Theorem Let $\mu: R \to E$ be a finitely additive measure. Then μ is strongly bounded on R if and only if, for each continuous semi-norm p, μ has a bounded, finitely additive p-control λ_p. Additionally, λ_p may be chosen so that:

(1) $\lambda_p(r) \leq \sup\{p(\mu(r')): r' \subseteq r, r \in R\}$;

(2) μ is countably additive if and only if every λ_p is countably additive.

Proof If μ' is the measure corresponding to μ on R_s, the Stone ring associated with R, then by Theorem 1.9(2) μ' is strongly bounded on R_s, so that Theorem 2.24 gives

us a λ_p' on R_s. The corresponding λ_p on R, then, is the bounded control desired. Conversely if each λ_p exists, then since it is bounded, it is strongly bounded, and one may prove μ strongly bounded as in Theorem 2.20. As for (2), if μ is countably additive, Theorem 2.24 shows that the countably additive λ_p exist, while if they do exist, then for $(r_i) \subseteq R$ pair-wise-disjoint,

$$\mu(U_j r_j) - \sum_{j=1}^{i} \mu(U_{j \geq i}\ r_j) \vec{i}\ 0,$$

since for each continuous semi-norm p,

$$\lambda_p(U_{j \geq i}\ r_j) = \lambda_p(U_j r_j) - \sum_{j=1}^{i} \lambda_p(r_j) \vec{i}\ 0$$

from the countable additivity of λ_p and the fact that λ_p is finite-valued. Finally we need to prove (1). Let p be any continuous semi-norm. If (1) is satisfied on R_s, then it is obviously satisfied on R also, so we assume μ to be countably additive. We found λ_p in Theorem 2.24 by extending μ to $\bar{\mu}$ on D, the δ-ring generated by R, and using Theorem 2.20, so that in the notation of Theorem 2.20,

$$\lambda_p(d) = \lambda(d) = \sum_i \frac{1}{2i}\ \lambda_i(d \cap d_i).$$

Since by Theorem 1.15, (1) holds for each λ_i on D/d_i,

we see that (1) also holds for λ_p on D, ie

$$\lambda_p(d) \leq \sup\{p(\bar{\mu}(d')): d' \subseteq d,\ d\in D\}, \qquad d\in D.$$

Denote this supremum by K. Choose $\epsilon > 0$, and let U be a balanced neighborhood of 0 such that $2U \subseteq V$, where

$$V = \{e\in E: p(e) \leq \epsilon\}.$$

Fix $r\in R$ and let d be any element of D such that $d \subseteq r$. By the definition of $\bar{\mu}$ on D, there is a sequence $(s_i) \subseteq R$ such that $\cup_i s_i \subseteq r$ and

$$\bar{\mu}(d) - \bar{\mu}(\cup_i s_i) \in U.$$

Then

$$\lim_j \mu(\cup_{i=1}^{j} s_i) = \bar{\mu}(\cup_i s_i) = \bar{\mu}(d) + \bar{\mu}(\cup_i s_i) - \bar{\mu}(d)$$

$$\in \bar{\mu}(d) + U,$$

so that for some $N\in\omega$,

$$\mu(\cup_{i=1}^{N} s_i) \in \bar{\mu}(d) + 2U \subseteq \bar{\mu}(d) + V.$$

But then, if we let $r' = \cup_{i=1}^{N} s_i \in R$, we have $r' \subseteq r$ while

$$\begin{aligned} p(\mu(r')) &= p(\mu(r') - \bar{\mu}(d) + \bar{\mu}(d)) \\ &\geq p(\bar{\mu}(d)) - p(\mu(r') - \bar{\mu}(d)) \\ &\geq p(\bar{\mu}(d)) - \epsilon. \end{aligned}$$

We see, then, that for $r \in R$,

$$\sup\{p(\mu(r')): r' \in R,\ r' \subseteq r\} = \sup\{p(\bar{\mu}(d)): d \in D,\ d \subseteq r\},$$

and hence that (1) is satisfied. □

2.26 Corollary. Let μ be as in the theorem but with E a Banach space. Then μ is strongly bounded if and only if there exists a bounded, finitely additive measure $\lambda \geq 0$ with $\mu \ll \lambda$. Additionally λ may be chosen so that

(1) $\lambda(r) \leq \sup\{||\mu(r')||: r' \in R,\ r' \subseteq R\}$, $r \in R$;

(2) μ is countably additive if and only if λ is countably additive.

2.27 Corollary. For a countably additive measure $\mu: R \to E$, the following are equivalent.

(1) μ is strongly bounded

(2) μ extends to a countably additive measure $\bar{\mu}$ from $\sum(R)$ into $\tilde{E}$.

(3) for each continuous semi-norm p, μ has a bounded p-control λ_p.

The corollary just above shows that in the bounded measure case the Bartle-Dunford-Schwartz Theorem remains valid on a ring precisely when the vector measure is strongly bounded. This problem (for δ-rings) was raised

by Dinculeanu and Kluvanek [5]. Of course Theorem 2.20 suffices to solve this problem for a δ-ring.

The p-control, λ_p, whose existence is guaranteed in Theorem 2.25 for a strongly bounded μ, may fail to exist if we assume only that μ is bounded, even if λ_p is allowed to be unbounded and R is required to be a σ-algebra. Even if μ is countably additive, λ_p may not exist when R is a ring. Examples are given in (1) and (2) below.

2.28 Example. (1) Let L be the σ-algebra of Lebesque measurable subsets of $[0, 1]$, and let $\mathcal{L}_\infty(L)$ be the Banach space of measurable functions on Ω, with the supremum norm

$$||f||_\infty = \sup\{|f(x)|: x\in[0, 1]\}.$$

Define $\mu: L \to \mathcal{L}_\infty(L)$ by $\mu(r) = \xi_r$, $r\in L$. Let $I = \{\frac{1}{i}: i \geq 1, i\in\omega\}$. If λ is a $||\cdot||_\infty$-control for μ, and $\delta > 0$ is such that $\lambda(r) < \delta$ implies $||\mu(r)|| < 1$, then $\lambda(\{\frac{1}{i}\}) \geq \delta$, $i\in I$. Since $I \in L$,

$$\lambda(I) \geq \lambda((\tfrac{1}{i})_{i=1}^{n}) \geq n\delta \to \infty,$$

shows that $\lambda(I)$ cannot be finite.

(2) Let R be the ring generated by $I\cup\{I\}$. Then $\mu_{/R}$ is trivially countably additive. If

λ is a control for μ, then $\lambda(I) = \infty$ as before.

2.29 Remark. If $\mu: R \to E$ is a finitely additive measure, then μ is strongly bounded if and only if the range of μ is weakly sequentially compact. That this is a consequence of Theorem 2.25 has been noted in Brooks [2].

It is natural to wonder in view of the Nikodym Theorem (see Dunford and Schwartz [6], p. 321) whether the corresponding theorem is true for finitely additive, strongly bounded measures. For a σ-algebra, the next theorem answers this affirmatively, and the following one gives a condition for it to be true on a ring. The first theorem was mentioned in Brooks and Jewitt [3], but we give it as a nice application of the existence of a control measure.

2.30 Theorem. Let $\mu_i: R \to E$, $i\in\omega$, be strongly bounded, finitely additive measures on a σ-algebra R. If $\lim_i \mu_i(r)$ exists in $\widetilde{E}$, $r\in R$, then $\mu(r) = \lim_i \mu_i(r)$, $r\in R$, defines a strongly bounded, finitely additive measure μ.

Proof Let p be a continuous semi-norm on E. By Theorem 2.25, each μ_i has a bounded p-control λ_i. Then

$$\lambda(r) = \sum_i \frac{1}{2^i} \frac{\lambda_i(r)}{1+B_i}, \qquad r\in R,$$

where B_i is a bound on λ_i, defines a bounded, finitely additive measure on R such that $\lim_{\lambda(r)\to 0} \mu_i(r)=0$, $i\in\omega$. Then by Theorem 3 of Brooks and Jewitt [3], the limit above is uniform in i, so that μ is strongly bounded since λ is bounded. □

2.31 Theorem. Let $\mu_i: R \to E$, $i\in\omega$, be strongly bounded, finitely additive measures. If (μ_i) is uniformly strongly bounded, and if $\lim_i \mu_i(r)$ exists in $\tilde{E}$, $r\in R$, then $\mu(r) = \lim_i \mu_i(r)$, $r\in R$, defines a strongly bounded, finitely additive measure.

Proof The measure μ is finitely additive since each μ_i is finitely additive. Suppose (r_j) is pair-wise-disjoint in R. Then if p is a continuous semi-norm on E, there exists an N such that for all i and all $j \geq N$,

$$p(\mu_i(r_j)) \leq 1.$$

But p has a unique extension to a continuous semi-norm $\tilde{p}$ on $\tilde{E}$, so that

$$\tilde{p}(\mu(r_j)) = \lim_i p(\mu_i(r_j)) \leq 1,$$

for all $j \geq N$. Thus μ is also strongly bounded. □

2.32 Definition. Let $\mu: R \to E$ be a finitely additive measure. Then μ is locally strongly bounded if, for each $r \in R$, μ restricted to the algebra $R/_r$ is strongly bounded.

As we see in the next theorem the condition defined above is a generalization of strong boundedness which reduces to strong boundedness on a σ-ring.

2.33 Theorem. If $\mu: R \to E$ is a finitely additive measure, then μ is locally strongly bounded if and only if (r_i) monotone decreasing in R implies $\lim_i \mu(r_i)$ exists in $\tilde{E}$.

Proof ($\Rightarrow$) If (r_i) is monotone decreasing, then $(r_i) \subseteq R/_{r_1}$, and μ is strongly bounded on $R/_{r_1}$, so that the limit exists.

($\Leftarrow$) Let $r \in R$, and $(r_i) \subseteq R/_r$ monotone increasing. Then $(r-r_i)_i$ is monotone decreasing so that

$$\lim_i \mu(r_i) = \lim_i [\mu(r) - (\mu(r-r_i))]$$

$$= \mu(r) - \lim_i \mu(r-r_i)$$

exists in $\tilde{E}$. □

2.34 Corollary. If R is a σ-ring, then μ is locally strongly bounded if and only if μ is strongly bounded.

Proof By Corollary 2.3, μ is necessarily locally strongly bounded when it is strongly bounded. Conversely if (r_i) is monotone increasing in R, then $(\bigcup_j r_j - r_i)_i$ is monotone decreasing so that as in the theorem above, $\lim_i \mu(r_i)$ exists in $\tilde{E}$. □

2.35 Theorem. Let $\mu: R \to E$ be a finitely additive measure. If, for each continuous semi-norm p, μ has a p-control λ_p, then μ is locally strongly bounded.

Proof If $r \in R$, then each λ_p is bounded on $R/r\backslash$ That μ is then strongly bounded on R/r follows as in Theorem 2.20. □

2.36 Remark. Local strong boundedness is actually weaker than strong boundedness since the measure in Example 2.19 satisfies the first, but not the second condition. By Corollary of Dinculeanu and Kluvanek [5], a locally strongly bounded countably additive $\mu: R \to E$ can be extended to a countably additive measure $\bar{\mu}: D \to \tilde{E}$, where D is the δ-ring generated by R. A proof of this is also contained in the proof of Theorem 2.24.

CHAPTER III

Uniform Strong Boundedness And Weak Sequential Compactness In ba(Ω, A)

This chapter is basically concerned with two problems. The first is that of extending conditions holding on a ring R onto the σ-ring $\sum(R)$. A basic tool for this will be the extension theorem of Chapter 2, Theorem 2.24, although most of the results of Chapter 2 will be needed. The second problem is that of investigating conditions for weak sequential compactness in ba(Ω, A), the space of bounded, finitely additive, scalar-valued measures on an algebra A with the variation norm:

$$||\mu|| = \nu(\mu)(\Omega).$$

Throughout this chapter A will denote an algebra, and $\sum$ will denote the σ-ring generated by R.

3.1 Theorem. If $\mu: R \to \Re$ is a bounded finitely additive measure, then μ is countably additive if and only if μ^+ and μ^- are countably additive.

Proof ($\Leftarrow$) Since $\mu = \mu^+ - \mu^-$, μ is obviously countably additive.

($\Rightarrow$) If $(r_i) \subseteq R$ is pair-wise-disjoint, it always follows that

$$\mu^+(\cup_i r_i) \geq \sum_i \mu^+(r_i),$$

since $\mu^+(\cup_i r_i) \geq \mu^+(\cup_{i=1}^n r_i) = \sum_{i=1}^n \mu^+(r_i)$, $n \in \omega$. But since μ is countably additive,

$$\begin{aligned}
\mu^+(\cup_i r_i) &= \sup\{\mu(s): s \in R,\ s \subseteq \cup_i r_i\} \\
&= \sup\{\textstyle\sum_i \mu(s_i): (s_i) \subseteq R,\ \cup_i s_i \in R,\ s_i \subseteq r_i,\ i \in \omega\} \\
&\leq \sup\{\textstyle\sum_i \mu^+(s_i): (s_i) \subseteq R,\ \cup_i s_i \in R,\ s_i \subseteq r_i,\ i \in \omega\} \\
&= \textstyle\sum_i \mu^+(r_i). \quad \square
\end{aligned}$$

Although we have not needed the fact so far, it is true that the extensions of measures we dealt with in Chapter 2 are unique. Since we shall implicitly use this several times in this chapter, we give a short proof.

3.2 Theorem. Let $\mu_i: \sum \to E$, $i = 1, 2$, be countably additive measures. If $\mu_1 = \mu_2$ on R, then $\mu_1 = \mu_2$.

Proof Let $F = \{s \in \sum: \mu_1(s) = \mu_2(s)\}$; $F \supseteq R$. If (f_j) is a monotone sequence in F, then since μ_i, $i = 1, 2$, are strongly bounded,

$$\lim_j \mu_1(f_j) = \lim_j \mu_2(f_j)$$

exists in $\tilde{E}$. Since μ_i, $i = 1, 2$ are countably additive

into E, the limits exist in E and

$$\mu_i(\lim_j f_j) = \lim_j \mu_i(f_j),$$

for i = 1, 2. Thus $\lim_j f_j \in F$. By the Theorem on Monotone Classes (Halmos [9]), we have that $F = \sum$. □

Our immediate goal, and the key theorem of this chapter is Theorem 3.6, but first we shall prove a special case to which it can be reduced using Theorems 3.1 and 2.18.

3.3 Lemma. Let $\mu_i: \sum \to \mathbb{R}^+$, $i \in \omega$, be countably additive measures. If $N \in \omega$, $\epsilon > 0$, $s \in \sum$, and (b_i) pair-wise-disjoint in R with $\bigcup_i b_i \supseteq s$, then there exists $(a_i) \subseteq R$ pair-wise-disjoint in R such that

(1) $\bigcup_i b_i \supseteq \bigcup_i a_i \supseteq s$,

and (2) $\sum_i \mu_n(a_i) < \mu_n(s) + \epsilon$, $0 \leq n \leq N$.

Proof Since each μ_i is an extension of itself from R to $\sum$, we may use Corollary 1.6 to find $(r_i') \subseteq R$ disjoint such that $\bigcup_i r_i' \supseteq s$ and $\sum_i \mu_1(r_i) < \mu_1(s) + \epsilon$. Then if we let (r_i) be the set $\{r_n' \cap b_m : n, m \in \omega\}$, we have a disjoint sequence $(r_i) \subseteq R$ such that $\bigcup_i b_i \supseteq \bigcup_i r_i \supseteq s$. Also

$$\sum_i \mu_0(r_i) = \sum_i \sum_j \mu_0(r_i' \cap b_j)$$

$$= \sum_i \mu_0(\cup_j (r_i' \cap b_j))$$

$$\leq \sum_i \mu_0(r_i') < \mu_0(s) + \epsilon.$$

So we have shown the lemma for N = 0. Suppose the lemma is valid for N-1. By Corollary 1.6 again, there exists $(d_i) \subseteq R$ disjoint such that $\cup_i d_i \supseteq s$ and $\sum_i \mu_N(d_i) < \mu_N(s) + \epsilon$. Also we have $(c_i) \subseteq R$ disjoint such that $\cup_i b_i \supseteq \cup_i c_i \supseteq s$, and for $1 \leq n \leq N-1$, $\sum_i \mu_n(c_i) < \mu_n(s) + \epsilon$. If we let (a_i) be the double sequence $(c_i \cap d_j)_{i,j \in \omega}$, then we have $\cup_i b_i \supseteq \cup_i c_i \supseteq \cup_i \cup_j (c_i \cap d_j) = \cup_i a_i \supseteq E$, while (a_i) is obviously disjoint in R. Also, for $0 \leq n \leq N-1$,

$$\sum_i \mu_n(a_i) = \sum_{i,j} \mu_n(c_i \cap d_j)$$

$$= \sum_i \mu_n(\cup_j (c_i \cap d_j))$$

$$\leq \sum_i \mu_n(c_i) < \mu_n(s) + \epsilon,$$

while

$$\sum_i \mu_N(a_i) = \sum_{i,j} \mu_N(c_i \cap d_j)$$

$$= \sum_j \mu_N(\cup_i (c_i \cap d_j))$$

$$\leq \sum_j \mu_N(d_j) < \mu_N(s) + \epsilon.$$

By induction, therefore, the lemma holds for $N \in \omega$. □

3.4 Theorem. Let $(\mu_\alpha)_{\alpha \in \mathfrak{A}}$, be a set of non-negative, finite-valued, countably additive measures on $\sum$. Then if (μ_α) is uniformly strongly bounded on R, (μ_α) is uniformly countably additive on $\sum$.

Proof Suppose (μ_α) is not uniformly countably additive on $\sum$. Then for some $\epsilon > 0$, there exists $(s_i) \subseteq \sum$, and $(\mu_i) \subseteq (\mu_\alpha)$, such that $s_i \downarrow \emptyset$, while $\mu_i(s_i) > \epsilon$. Fix $\epsilon_i = \epsilon/2^{i+2}$, $i \in \omega$, so that $\sum_i \epsilon_i = \epsilon/2$.

Since every $s \in \sum$ is covered by a countable union of elements of R, we can find by Lemma 3.3 a sequence $(r_i^0)_i \subseteq R$ such that $\bigcup_i r_i^0 \supseteq s_0$ and

$$\sum_i \mu_0(r_i^0) < \mu(s_0) + \epsilon_0.$$

Since (μ_α) is uniformly strongly bounded on R, there exists an $n_0 \in \omega$ such that

$$\sum_{i>n_0} \mu_j(r_i^0) < \alpha_0, \qquad j \in \omega.$$

Define $r_0 = \bigcup_{i=1}^{n_0} r_i^0$. Then for $j \in \omega$,

$$\mu_j(r_0 \cap s_j) = \mu_j(s_j) - \mu_j(s_j - r_0)$$

$$\geq \mu_j(s_j) - \sum_{i>n_0} \mu_j(r_i^0)$$

$$\geq \epsilon - \epsilon_0.$$

Thus we have for n = 0, a decreasing finite sequence $(r_i)_{i=0}^n \subseteq R$ such that

(1) $\mu_j(r_i) < \mu_j(s_i) + \epsilon_i, \qquad 0 \leq j \leq i,$

(2) $\mu_j(r_i \cap s_j) \geq \epsilon - \sum_{k=0}^{i} \epsilon_k, \qquad j \geq i,$

for $0 \leq i \leq n$. If we have (1) and (2) for n = m, we can find by Lemma 3.3 (with $b_0 = r_m$, $b_i = \emptyset$, $i > 0$) a sequence (r_i^{m+1}) disjoint in R such that $r_m \supseteq \cup_i r_i^{m+1} \supseteq s_{m+1} \cap r_m$

$$\sum_i \mu_j(r_i^{m+1}) < \mu_j(s_{m+1}) + \epsilon_{m+1},$$

for $0 \leq j \leq m+1$. Since (μ_α) is uniformly strongly bounded on R, there is an n_{m+1} such that for $i \in \omega$

$$\sum_{j>n_{m+1}} \mu_i(r_j^{m+1}) < \epsilon_{m+1}.$$

Then defining $r_{m+1} = \cup_{j=1}^{n_{m+1}} r_j^{n+1}$, we have (1) and (2) above holding for n = m+1, since for $j \geq m+1$

$$\mu_j(r_{m+1} \cap s_j) = \mu_j(s_j \cap r_m) - \mu_j((s_j \cap r_m) - r_{m+1})$$

$$\geq \epsilon - \sum_{j=0}^{m} \epsilon_j - \sum_{j>n_{m+1}} \mu_j(r_j^{m+1})$$

$$\geq \epsilon - \sum_{j=0}^{m+1} \epsilon_j.$$

Thus by induction (1) and (2) hold for all $i\epsilon\omega$ with the decreasing sequence (r_i). Then by (2)

$$\mu_i(r_i) \geq \mu_i(r_i \cap s_i)$$

$$\geq \epsilon - \sum_{k=0}^{i} \epsilon_k \geq \epsilon/2.$$

Since (μ_i) is uniformly strongly bounded, we have by Theorem 2.11 that $\lim_j \mu_i(r_j)$ exists uniformly in i, so that by the above inequality $\lim_j \mu_i(r_j) \geq \epsilon/2$, $i\epsilon\omega$. But this contradicts the fact that $\lim_j \mu_i(r_j) = 0$, $i\epsilon\omega$, by (1) above. □

The proof of Theorem 3.4 above was greatly simplified by Dr. Stephen A. Saxon. He has also made many helpful suggestions incorporated throughout this dissertation.

3.5 Remark. Although uniform countable additivity is a weaker concept than uniform strong boundedness (when the measures are countably additive), if R is a σ-ring, the

two concepts are the same. Thus in Theorem 3.6, uniform countable additivity could be replaced by uniform strong boundedness (or uniform additivity, by Theorem 2.11).

3.8 Theorem. Let $\mu_\alpha: R \to E$, $\alpha \in \mathfrak{A}$, be countably additive measures. If (μ_α) is uniformly strongly bounded on R, then each μ_α has a countably additive extension $\bar{\mu}_\alpha$ from $\sum$ into $\tilde{E}$ such that $(\bar{\mu}_\alpha)$ is uniformly countably additive.

Proof Since (μ_α) is uniformly strongly bounded, each μ_α is strongly bounded, so that $\bar{\mu}_\alpha$ exists by Theorem 2.22. Suppose that $(\bar{\mu}_\alpha)$ is not uniformly countably additive on $\angle$. Then we can find some $(s_i) \subseteq \sum$ and some $(\mu_i) \subseteq (\mu_\alpha)$ such that $s_i \downarrow^i \phi$, while for some $\epsilon > 0$, and some continuous semi-norm p,

$$p(\mu_i(s_i)) > \epsilon, \qquad i \in \omega.$$

Define $X = \{x \in E' : |x(e)| \leq p(e),\ e \in E\}$. Since each $\bar{\mu}_\alpha$ is countably additive on $\sum$, so is $x\mu_\alpha$, $x \in X$. If $(r_i) \subseteq R$ pair-wise-disjoint, then by the uniform strong boundedness of (μ_α) on R, there is an $N \in \omega$ such that $i \geq N$, $\alpha \in \mathfrak{A}$ implies

$$p(\mu_\alpha(r_i)) < \epsilon.$$

Since $|x\mu_\alpha(r_i)| \leq p(\mu_\alpha(r_i))$, $x\in X$, we see that $(x\mu_\alpha)_{\alpha\in\mathfrak{A}}$, $x\in X$ is uniformly strongly bounded on R. By Theorem 2.18, then, so are $((x\mu_\alpha)^+)$ and $((x\mu_\alpha)^-)$ uniformly strongly bounded on R. Thus by Theorem 3.4 there exists $M\in\omega$ such that $i \geq M$ implies for $\alpha\in\mathfrak{A}$, $x\in X$,

$$(x\mu_\alpha)^+(s_i) < \epsilon/2,\ (x\mu_\alpha)^-(s_i) < \epsilon/2.$$

But by Theorem 1.3,

$$|x\mu_\alpha(s_i)| \leq \nu(x\mu_\alpha)(s_i) = (x\mu_\alpha)^+(s_i) + (x\mu_\alpha)^-(s_i) < \epsilon,$$

so we have a contradiction, since, for each $i\in\omega$, there is some $x\in X$ such that

$$|x\mu_i(s_i)| = p(\mu_i(s_i)) > \epsilon. \quad \square$$

One special case of Theorem 3.6 which is frequently useful in proofs by contradiction is given in the following corollary.

3.7 Corollary. Let the μ_α be defined on $\sum$, and let $[s_i]$ be the ring generated by $(s_i)_i \subseteq \sum$. Then if (μ_α) is uniformly strongly bounded on $[s_i]$, (μ_α) is uniformly countably additive on $\sum([s_i])$.

The next theorem is a very useful extension of a result of Dunford and Schwartz [6], p. 292.

3.8 Theorem. Let $\mu_i: R \to E$, $i\in\omega$, be countably additive measures. If (μ_i) is uniformly strongly bounded, and if $\lim_i \mu_i(r)$ exists in $\widetilde{E}$, $r\in R$, then $\lim_i \bar{\mu}_i(s)$ also exists in $\widetilde{E}$, $s\in\sum$, where $\bar{\mu}_i$ is the extension of μ_i to $\sum$.

Proof Define $L = \{s\in\sum: \lim_i \bar{\mu}_i(s) \text{ exists in } \widetilde{E}\}$. We have by hypothesis that $R \subseteq L$, so that if we show L to be a monotone class, the Theorem on Monotone Classes (Halmos [9]) will give us that $L = \sum$, i.e. that $\lim_i \bar{\mu}_i(s)$ exists on $\sum$. Suppose that $(s_i) \subseteq L$ is monotone increasing, and let $s = \lim_i s_i$. Since by Theorem 3.6, $(\bar{\mu}_i)$ is uniformly countably additive on $\sum$, there exists an $N\in\omega$ such that

$$\mu_i(s-s_N) \in V, \qquad i\in\omega,$$

where V is an arbitrary circled neighborhood of 0. Since $\lim_i \mu_i(s_N)$ exists in $\widetilde{E}$, there is an $M\in\omega$ such that for $i, j \geq M$,

$$\mu_i(s_N) - \mu_j(s_N) \in V.$$

Thus for $i, j \geq M$,

$$\mu_i(s) - \mu_j(s) = \mu_i(s-s_N) + \mu_i(s_N) - \mu_j(s_N) - \mu_j(s-s_N)$$

$$\in V + V + V = 3V.$$

Since for any neighborhood U of 0, there is a circled neighborhood of 0, say V, such that $3V \subseteq U$, we see that $\lim_i \mu_i(r)$ exists in $\widetilde{E}$. If $(s_i) \subseteq L$ is monotone decreasing, then

$$\lim_i \mu_i(s) = \lim_i [-\mu_i(s_1-s) + \mu_i(s_1)]$$

exists in $\widetilde{E}$ since $(s_1-s_i) \uparrow s_1-s$. We have $s_1-s_i \in L$, $i\in\omega$, since for each i,

$$\lim_j \mu_j(s_1-s_i) = \lim_j [\mu_j(s_1) - \mu_j(s_i)]$$

exists in $\widetilde{E}$. □

3.9 Corollary Suppose $\lim_i \mu_i(r)$ exists in $\widetilde{E}$, $r\in R$. Then (μ_i) is uniformly strongly bounded on R if and only if each μ_i has a countably additive extension $\bar{\mu}_i$ to $\sum$ such that $\lim_i \bar{\mu}_i(s)$ exists in $\widetilde{E}$, $s\in\sum$.

Proof The necessity is Theorem 3.8. Conversely, if the limits exist on $\sum$, then by the Vitali-Hahn, Saks Theorem (Dunford and Schwartz [6]) the $(\bar{\mu}_i)$ are uniformly countably additive on $\sum$, equivalently uniformly strongly

bounded on $\sum$. But then (μ_i) is uniformly strongly bounded on R. □

3.10 Theorem. Let (μ_α) be a set of finitely additive measures on R into E. If, for each continuous semi-norm p, there is a finitely additive, bounded measure λ_p on R into $\mathfrak{R}$ such that $\mu_\alpha <<_p \lambda_p$, $\alpha\in\mathfrak{A}$, and if (μ_α) is point-wise bounded, then the following are equivalent:

(1) (μ_α) is uniformly strongly bounded;

(2) $\mu_\alpha <<_p \lambda_p$, uniformly in α, for each continuous semi-norm p.

Proof (2 → 1) A basic neighborhood of 0 in E is of the form $\{e\in E: p_i(e) \le 1, 1\le i\le n\}$, where each p_i is a continuous semi-norm. Let (r_j) be pair-wise-disjoint in R. Then for $1 \le i \le n$ there exists an $N_i\in\omega$ such that $j \ge N_i$ implies $\lambda_{p_i}(r_j) < \delta(1)$. If $N = \max\{N_i: 1\le i\le n\}$, then $j \ge N$ implies $p_i(\mu_\alpha(r_j)) \le 1$, $\alpha\in\mathfrak{A}$, $1\le i\le n$. Thus $\mu_\alpha(r_j) \overset{j}{\to} 0$ uniformly in α.

(1 → 2) Suppose that for some continuous semi-norm p, $\mu_\alpha <<_p \lambda_p$ does not hold uniformly in α. Then there exists $(r_i) \subseteq R$ and $(\mu_i) \subseteq (\mu_\alpha)$ such that for some $\epsilon > 0$,

$$\lambda_p(r_i) < \frac{1}{2^i}\ ,\ p(\mu_i(r_i)) > \epsilon, \qquad i\epsilon\omega.$$

For each $i\epsilon\omega$, let $x_i\epsilon E'$ such that

$$|x_i\mu_i(r_i)| = p(\mu_i(r_i)).$$

and $$|x_i(e)| \leq p(e), \qquad e\epsilon E.$$

Since (μ_i) is point-wise bounded, we may use a Cantor diagonal process to find a subsequence $(x_{n_j})_j$ of $(x_i)_i$ such that for $r\epsilon[r_i]$

$$\lim_j x_{n_j}\mu_{n_j}(r)$$

exists and is finite, for each $i\epsilon\omega$. Without loss of generality, we may assume the limits exist and are finite for the original sequence $(x_i\mu_i)$. The reader can easily see from $|x_i(e)| \leq p(e)$, $e\epsilon E$, that for each $i\epsilon\omega$, $x_i\mu_i << \lambda_p$. In addition, we will show that the sequence $(x_i\mu_i)$ is uniformly strongly bounded. Suppose $(s_i) \subseteq R$ is pair-wise-disjoint. Then since $\mu_\alpha(s_i) \overset{i}{\to} 0$ uniformly in α, there exists an $N\epsilon\omega$ such that for $i \geq N$, $p(\mu_j(s_i)) < \epsilon$, for an arbitrary fixed $\epsilon > 0$, and $j\epsilon\omega$. Then obviously, for $i \geq N$, $|x_j\mu_j(s_i)| < \epsilon$, $j\epsilon\omega$.

We may now assume that all measures under consideration are countably additive, since otherwise we could go over to the Stone ring associated with R without changing any relationships used below.

We may now apply Theorem 3.8, so that

$$\lim_i \overline{x_i\mu_i}\ (s) < \infty$$

for $s\in\sum([r_i])$. Also, by Theorem 2.23, for $i\in\omega$, $\overline{x_i\mu_i} \ll \bar{\lambda}_p$, where, of course, $\bar{\lambda}_p$ is the countably additive extension of λ_p to $\sum([r_i])$ into $\mathfrak{R}$. By the Vitali-Hahns-Saks Theorem (Dunford and Schwartz [6]), we have $\overline{x_i\mu_i} \ll \bar{\lambda}_p$ uniformly in $i\in\omega$. Thus there exists a $\delta > 0$ such that $s\in\sum([r_i])$, $\bar{\lambda}_p(s) < \delta$ implies $|\overline{x_i\mu_i}(s)| < \epsilon$. Choose $i\in\omega$ such that $\frac{1}{2^i} < \epsilon$. Then by the choice of (r_i), $\lambda_p(r_i) < \delta$, but

$$|x_i\mu_i(r_i)| = p(\mu_i(r_i)) > \epsilon,$$

so that we have arrived at a contradiction. □

3.11 Corollary. Let $\mu_\alpha\colon R \to \mathfrak{X}$, $\alpha\in\mathfrak{A}$, be finitely additive measures. If there exists a bounded, finitely additive measure λ such that $\mu_\alpha \ll \lambda$, $\alpha\in\mathfrak{A}$, then the following are

equivalent when (μ_α) is point-wise bounded:

(1) (μ_α) is uniformly strongly bounded;

(2) $\mu_\alpha << \lambda$, uniformly in α.

Proof The proof of Theorem 3.10 obvicusly holds if p is restricted to any family of continuous semi-norms which generated the topology on E. □

3.12 Corollary. Let $\mu_i: R \to \mathfrak{X}$, $i\in\omega$, be finitely additive measures. If (μ_i) is point-wise bounded, the following are equivalent:

(1) (μ_i) is uniformly strongly bounded;

(2) there exists a bounded, finitely additive $\lambda: R \to \mathfrak{R}$ such that $\mu_i << \lambda$ uniformly in i.

When each μ_i is countably additive, λ may be chosen countably additive.

Proof We need only show (1 → 2). Since each μ_i is strongly bounded, there exists, for $i\in\omega$, a bounded control measure λ_i for μ_i. Define

$$\lambda(r) = \sum_i \frac{1}{2^i} \frac{\lambda_i(r)}{1 + B_i} ,$$

where B_i is a bound for λ_i. Then $\lambda: R \to \mathfrak{R}$ is a bounded,

finitely additive measure such that $\mu_i << \lambda$, $i \in \omega$. Thus Theorem 3.10 gives the uniformity in i. If each μ_i is countably additive, λ may be made countably additive by choosing each λ_i to be countably additive. □

3.13 Theorem. Let $\mu_\alpha: R \to E$, $\alpha \in \mathfrak{A}$ be countably additive measures. If (μ_α) is point-wise bounded, and if, for each continuous semi-norm p, there is a bounded, countably additive measure $\lambda_p: R \to \mathbb{R}$ such that $\mu_\alpha <<_p \lambda_p$, uniformly in α, then

$$\bar{\mu}_\alpha <<_p \bar{\lambda}_p, \text{ uniformly in } \alpha,$$

where $\bar{\mu}_\alpha$, $\bar{\lambda}_p$ are the extensions of μ_α, λ_p to $\sum$.

Proof By Theorem 2.23, $\bar{\mu}_\alpha <<_p \bar{\lambda}_p$, for each α. By Theorem 3.10, (μ_α) is uniformly strongly bounded on R, hence also $(\bar{\mu}_\alpha)$ on $\sum$ by Theorem 3.6. Since (μ_α) is point-wise bounded and uniformly strongly bounded on R, Theorem 2.16 gives that (μ_α) is bounded on R, and hence $(\bar{\mu}_\alpha)$ is bounded on $\sum$ by Theorem 2.23. Then applying Theorem 3.10 again, we have that $\bar{\mu}_\alpha <<_p \lambda_p$, uniformly in $\alpha \in \mathfrak{A}$. □

The next theorem could actually be proved for an arbitrary space E, but for simplicity we do not do so.

3.14 Theorem. Let $\mu_\alpha: R \to X$, $\alpha \in G$, be finitely additive measures. Then the following are equivalent:

(1) (μ_α) is uniformly strongly bounded;

(2) there exists a bounded, finitely additive $\lambda: R \to \mathbb{R}$ such that $\mu_\alpha << \lambda$ uniformly in α.

Proof We proved (2 → 1) in Theorem 3.10. To show (1 → 2), we may first assume that each μ_α is countably additive, since the (μ_α) on the Stone ring corresponding to R would still be uniformly countably additive, and if (2) were satisfied on the Stone ring, it would also hold on R. It is enough to prove (1 → 2) with R replaced by $\sum$, since by Theorem 3.6 (1) holds for $(\bar{\mu}_\alpha)$ on $\sum$ and (2) holding on $\sum$ trivially implies (2) holding on R. Thus (μ_α) is a set of uniformly countably additive measures on $\sum$ into X. Define $(\lambda_\beta)_{\beta \in \mathcal{B}} = \{x'\mu_\alpha: \alpha \in G, x' \in X', ||x'|| \leq 1\}$. Then $(\lambda_\beta) \subseteq ca(\Omega, \sum)$ is uniformly countably additive, so that by Theorem 1.14, there exists a countably additive $\lambda: \sum \to \mathbb{R}^+$ with $\lambda_\beta << \lambda$ uniformly in β. Thus $\epsilon > 0$ implies there is a $\delta > 0$ such that if $\lambda(s) < \delta$ then

$$||\mu_\alpha(s)|| = \sup\{x'\mu_\alpha(s): |x'| \leq 1, x' \in X'\}$$
$$\leq \epsilon.$$

Since δ is independent of α, $\mu_\alpha \ll \lambda$ uniformly in α. □

3.15 Remark. The rest of this chapter is devoted to investigating weak sequential compactness in ba(Ω, A) using the machinery developed so far. From now on we take R = A, so that $\sum = \sum(A)$. We recall that by the Eberlein-Šmulian Theorem (Dunford and Schwartz [6], p. 430) a set K in a Banach space is weakly sequentially compact if and only if the weak closure of K is weakly compact.

3.16 Theorem. Let (μ_α) be a subset of ba(Ω, A). Then (μ_α) is weakly sequentially compact if and only if

(1) (μ_α) is point-wise bounded,

and (2) (μ_α) is uniformly strongly bounded.

Proof We shall first show that (1) and (2) imply (μ_α) is weakly sequentially compact. As in Theorem 1.9 (3), $(T\mu_\alpha) \subseteq ca(\Omega_s, A_s)$ is uniformly strongly bounded; obviously, $(T\mu_\alpha)$ is point-wise bounded. Thus by Theorem 3.6, $(\overline{T\mu_\alpha})$ is uniformly countably additive on $\sum$, while by Theorem 2.16, $(\overline{T\mu_\alpha})$ is bounded. Thus $(\overline{T\mu_\alpha}) \subseteq ca(\Omega_s, \sum(A_s))$ is weakly sequentially compact by Theorem 1.13. Since T is an isomorphism, if $(\mu_i) \subseteq (\mu_\alpha)$, there exists a sub-

sequence $(T\mu_{n_i})_i$ of $(\overline{T\mu_i})$ and a $\mu \in ca(\Omega_s, \sum(A_s))$ such that $\overline{T\mu_{n_i}}$ converges weakly to μ. Since $ca(\Omega_s, \sum(A_s))$ is a closed linear subspace of $ca(\Omega_s, A_s)$, we also have

$$T\mu_{n_i} \vec{w} \mu/_{A_s}$$

in $ca(\Omega_s, A_s)$. Since T is an isomorphism, we then have that

$$x(\mu_{n_i}) = xT^{-1}(T\mu_{n_i}) \vec{i} xT^{-1}(\mu/_{A_s}),$$

for each $x \in (ba(\Omega, A))'$, so that $\mu_{n_i} \vec{w} T^{-1}(\mu/_{A_s})$. And so we see that (μ_α) is weakly sequentially compact.

Conversely when (μ_α) is weakly sequentially compact, it is also bounded, so that (1) holds. Since T is an isomorphism, it follows much as above that $(\overline{T\mu_\alpha})$ is weakly sequentially compact. Thus by Theorem 1.13, $(\overline{T\mu_\alpha})$ is uniformly countably additive on $\sum(A_s)$, so that $(T\mu_\alpha)$ is uniformly strongly bounded on A_s. Thus (μ_α) is uniformly strongly bounded on A. □

3.17 Theorem. Let (μ_α) be a subset of $ca(\Omega, \sum)$. Then (μ_α) is weakly sequentially compact if and only if (μ_α)

is point-wise bounded on A, and there exists a countably additive $\lambda: A \to \mathbb{R}^+$ such that $\mu_\alpha << \lambda$ on A, uniformly in α.

Proof If (μ_α) is weakly sequentially compact, then by Theorem 3.16, (μ_α) is uniformly strongly bounded and point-wise bounded on A since $ca(\Omega, \sum)$ is a closed linear subspace of $ba(\Omega, A)$. Thus by Corollary 3.12 there exists $\lambda: A \to \mathbb{R}^+$ such that $\mu_\alpha << \lambda$ uniformly in α. As remarked, λ may be chosen countably additive.

Conversely, if λ does exist with $\mu_\alpha << \lambda$ on A, then (μ_α) is uniformly strongly bounded on A, so that by Theorem 2.16, (μ_α) is uniformly bounded. But then (μ_α) is uniformly bounded and uniformly strongly bounded on $\sum$ by Theorems 2.23 and 3.6 respectively, so that by Theorem 3.16, (μ_α) is weakly sequentially compact in $ba(\Omega, A)$. Since $ca(\Omega, A)$ is a closed subspace of $ba(\Omega, A)$, (μ_α) is weakly sequentially compact in $ca(\Omega, \sum)$. □

3.18 Convention. Let $(\mu_i)_{i\in\omega}$ be a sequence in $ba(\Omega, A)$. Then for $a\in A$,

$$\varphi(a) = \sum_i \frac{1}{2^i} \cdot \frac{\nu(\mu_i)(a)}{1+\nu(\mu_i)(\Omega)} ,$$

defines a bounded, finitely additive measure on A into

$\mathfrak{R}^+$ which we call the control of the sequence (μ_i).

3.19 Theorem. If $(\mu_\alpha) \subseteq ba(\Omega, A)$ is point-wise bounded, the following are equivalent:

(1) (μ_α) is weakly sequentially compact;

(2) (μ_α) is uniformly strongly bounded;

(3) if (a_i) is disjoint in A, then

$$\lim_i \nu(\mu_\alpha)(r_i) = 0,$$

uniformly in α;

(4) if (a_i) is disjoint in A, then

$$\sum_1 \nu(\mu_\alpha)(r_i)$$

converges in $\mathfrak{R}$, uniformly in α.

(5) for each sequence $(\mu_i) \subseteq (\mu_\alpha)$,

$$\mu_i << \varphi,$$

uniformly in i, where φ is the control of (μ_i).

Proof Of course (1 $\Leftrightarrow$ 2) by Theorem 3.15. By Theorem 3.10, a sequence $(\mu_i) \subseteq (\mu_\alpha)$ is uniformly strongly bounded if and only if $\mu_i << \varphi$ uniformly in i, so that (2 $\rightarrow$ 5) is clear, while (5 $\rightarrow$ 1) since by Theorem 3.15,

each subsequence (μ_i) of (μ_α) is then weakly sequentially compact. We shall show $(2 \to 3 \to 4 \to 2)$ to complete the proof. If (2) holds, then by (4) of Theorem 2.11, $\lim_{n,m} ||\mu_\alpha||(U_{i=n}^{m} r_i) = 0$ uniformly in α. Since by (2) and (3) of Theorem 1.3, $\nu(\mu_\alpha) \leq 2||\mu_\alpha||$, $\alpha, \in \mathfrak{A}$, (3) of the theorem follows. Actually we have proved more, in fact that

$$\lim_{m,n} \nu(\mu_\alpha)(U_{i=n}^{m} r_i) = 0,$$

uniformly in α. From this it is obvious that (4) holds. If (4) does hold, then $\nu(\mu_\alpha)(r_i) \underset{i}{\to} 0$ uniformly in α, so that $|\mu_\alpha(r_i)| \leq \nu(\mu_\alpha)(r_i)$ implies (μ_α) is uniformly strongly bounded. □

3.20 Remark. In (4) of Theorem 3.19,

$$\sum_i \nu(\mu_\alpha)(r_i)$$

may be replaced by

$$\sum_i |\mu_\alpha(r_i)|,$$

by Theorem 2.11 (2). The equivalence of this to (1) in Theorem 3.19 and the equivalence of (1) and (2) in Theorem 3.21 was done by Porcelli [13] in a very long and difficult manner.

3.21 Theorem. For $(\mu_i) \subseteq ba(\Omega, A)$, the following are equivalent:

(1) μ_i converges weakly to 0;

(2) (Porcelli) for (a_i) disjoint in A,

$$\lim_i \sum_j |\mu_i(a_j)| = 0;$$

(3) (Hildebrandt) for (a_i) disjoint in A,

$$\lim_i \sum_j \mu_i(a_j) = 0;$$

(4) a) (μ_i) is uniformly strongly bounded, and

b) $\lim_i \mu_i(a) = 0$, $a \in A$.

Proof (1 $\Leftrightarrow$ 4) This is Theorem 3.16, since (b) implies that (μ_i) is point-wise bounded and, if μ_i converges weakly, μ_i converges weakly to 0 if and only if (b) holds.

(2 $\rightarrow$ 3) is obvious.

(3 $\rightarrow$ 4): That (b) holds is seen by taking $a_1 = a$, $a_i = \phi$, $i > 1$. Suppose (a) fails. Then by Theorem 2.11 (4), there is a double sequence (m_i, n_i) such that $m_i \uparrow \infty$, $n_i \geq m_i$, $i \in \omega$, and for some $\epsilon > 0$, and some subsequence

(μ_{k_i}) of (μ_i),

$$\left|\sum_{j=m_i}^{n_i} \mu_{k_i}(a_j)\right| > \epsilon, \qquad i\epsilon\omega.$$

Clearly we may assume $m_{i+1} > n_i$, $i\epsilon\omega$. Then defining $b_i = \bigcup_{j=m_i}^{n_i} a_j$, $i\epsilon\omega$, we have (b_i) disjoint while $|\mu_{k_i}(b_i)| > \epsilon$, $i\epsilon\omega$. For simplicity, assume $|\mu_i(b_i)| > \epsilon$, $i\epsilon\omega$. Since for each $j\epsilon\omega$, there is a subsequence (b_{n_i}) of (b_i) such that $|\mu_{n_i}(b_{n_i})| = \mu_{n_i}(b_{n_i})$ for all $i \geq j$ or $|\mu_{n_i}(b_{n_i})| = -\mu_{n_i}(b_{n_i})$ for all $i \geq j$, we may assume by a diagonalization procedure that for each i,

$$|\mu_j(b_j)| = \mu_j(b_j),\ j \geq i, \text{ or } |\mu_j(b_j)| = -\mu_j(b_j),\ j \geq i.$$

Now since $\lim_i \mu_i(a) = 0$, $a\epsilon A$, we may also assume

$$\sum_{j=1}^{i-1} |\mu_i(b_j)| < \epsilon/2, \qquad i\epsilon\omega.$$

In fact this clearly holds for $i = 1$. Suppose we have the inequality for $i = N$. Then there exists an $M > N$ such that

$$|\mu_i(b_j)| < \frac{\epsilon}{2^{j+1}},$$

for $i \geq M$ and $1 \leq j \leq N$. If we relabel so that μ_i becomes

$\mu_{i-(M-N)+1}$, we have the inequality for $i = N + 1$. But now for all i,

$$|\sum_j \mu_i(b_j)| \geq |\sum_{j\geq i} \mu_i(b_j)| - \sum_{j=1}^{i-1} |\mu_i(b_j)|$$

$$\geq \sum_{j\geq i} |\mu_i(b_j)| - \epsilon/2$$

$$\geq \epsilon - \epsilon/2 = \epsilon/2,$$

using the fact that

$$|\sum_{j\geq i} \mu_i(b_j)| = \sum_{j\geq i} |\mu_i(b_j)|.$$

Since this clearly contradicts (3), we have shown (3 → 4).

(4 → 2): By Theorem 2.18, (μ_i^+) and (μ_i^-) are also uniformly strongly bounded. Thus, if $\epsilon > 0$, there exists an N such that for $i\epsilon\omega$

$$\sum_{j\geq N} |\mu_i(a_j)| \leq \sum_{j\geq N} (\mu_i^+(a_j)+\mu_i^-(a_j)) < \epsilon.$$

Also, from (b), there is an M such that $i \geq M$ implies

$$\sum_{j=1}^{N-1} |\mu_i(a_i)| < \epsilon.$$

Then for all $i \geq M$

$$\sum_j |\mu_i(a_j)| \leq \sum_{j\geq N} |\mu_i(a_j)| + \sum_{j=1}^{N-1} |\mu_i(a_j)|$$

$$< \epsilon + \epsilon = 2\epsilon,$$

so that (2) clearly holds. □

3.22 Remark. The proof of Theorem 3.21 contains an elementary proof of the equivalence of the conditions of Porcelli and Hildebrandt. Further, given Theorem 3.6, which is of interest in its own right, the proofs of Theorems 3.19 and 3.21 are relatively simple. Since it is of some interest to simplify the proof that Porcelli's condition implies weak convergence to 0 (see Darst [4]), a development of the theory of weak sequential compactness in $ba(\Omega, A)$ along the lines given in this chapter has an added bonus beyond the theorems discovered.

BIBLIOGRAPHY

[1] R. G. Bartle, N. Dunford, and J. Schwartz, "Weak compactness and vector measures," Canadian J. of Math.7(1955), 289-305.

[2] J. K. Brooks, "On the existence of a control measure," Bull. A.M.S. (to appear).

[3] J. K. Brooks and R. S. Jewitt, "On finitely additive vector measures," Proc. N.A.S. 67(1970), 3, 1294-1298.

[4] R. B. Darst, "A direct proof of Porcelli's condition for weak convergence," Proc. A.M.S. 17(1966), 1094-1096.

[5] N. Dinculeanu and I. Kluvanek, "On vector measures," Proc. London Math. Soc. (3)17(1967), 505-512.

[6] N. Dunford and R. Schwartz, Linear Operators, Part I, Interscience, New York, 1958.

[7] G. G. Gould, "Integration over vector-valued measures," Proc. London Math. Soc.(3)15(1965)193-225.

[8] A. Grothendieck, "Sur les applications linéares faiblement compactes d'espece du type C(K)," Canadian J. of Math., 5(1953), 129-173.

[9] P. R. Halmos, Measure Theory, D. Van Nostrand, Princeton, 1950.

[10] T. H. Hildebrandt, "On a theorem in the space ℓ_1 of absolutely convergent sequences with applications to completely additive set functions," Math. Research Center Rep. No. 62, Madison, Wisc., 1958.

[11] G. Köthe, Topological Vector Spaces, Springer-Verlag, New York, 1969.

[12] C. W. McArthur, "On a theorem of Orlicz and Pettis," Pacific J. of Math., 22(1967), 297-302.

[13] P. Porcelli, "Two embedding theorems with applications to weak convergence and compactness in spaces of additive type functions," J. Math. Mech. 9(1960), 273-292.

[14] C. E. Rickart, "Decompositions of additive set functions," Duke Math. J., 10(1943), 653-665.

[15] M. Sion, "Outer measures with values in a topological group," Proc. London Math. Soc. (3)19(1969), 89-106.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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